Marx's economics: a note on the transformation of values into prices and the law of the falling rate of profit

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Abstract

Following Marx, we consider a model with capital goods, wage goods and labour times. Actually, for simplicity we consider two capital goods and one wage good and labour times in these three sectors. The means of production are represented by a productive Leontief matrix. we obtain the following results

- We find the fondamental Marx result: the profit rate is strictly smaller than the exploitation rate. However, if we weaken the basic assumption of the Marx theory of exploitation then we obtain that the exploitation rate and the profit rate equal if, and only if, they are null. In some words, a capitalist economy makes profit in any sector if, and only if, there exists exploitation.
- Usually to transform the values into production prices the necessary and sufficient conditions are endogenous, in the sense they involve values and prices. In this paper, we exhibit exogenous necessary and sufficient conditions, i.e. involving only the technologies and the labour times.
- Regarding the falling rate of profit, Marx requires the exploitation rate be unchanged. In this paper, with a simple model, we show that the law of the falling rate holds while the exploitation rate changes.

Keywords: Values, labour-times, productive Leontief technology, exploitation rate, profit rate, production prices, transformation of values into production prices, the law of falling rate of profit.

JEL Classification: C62, C67, D57,D46

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1 Introduction

The literature on Marx's economic theory is extremely abundant. We cite some of them such as, of course Marx, [6] [7], Lange [4],Morishima[8], Romer[11], Robinson [10], Samuelson [13], Sweezy[14], Yaffe [15]. This paper is on the line of economists who use mathematics for Marx's economics. We cite some them, Dumenil and Levy [1], Foley [3], Morishima [8], Romer [11],[12].
This note is based on the chapters in Part I, Part II and Part III of the book "Marx's Economics" (Cambridge University Press, 1973) by Michio Morishima [8]
The quotations come from Karl Marx's *Capital*, Progress Publishers, Moscow (vol.1, 1965)[6].
We focus in this note on two issues.

- Necessary and sufficient conditions to transform values in production prices. The well-known conditions come from Marx himself

  - Total values= Total prices
  - Total surplus= total profits
  - Uniform profit rate

  Another endogenous necessary and sufficient condition is equality of the capital compositions in the sectors. It is given in Morishima [8]
  These conditions are "endogenous" in the sense they involve values and prices.
  In this note we give "exogenous" necessary and sufficient conditions to transform the values into prices. They are "exogenous" because they involve only the technology and the the labour-times.

- Law of falling rate of profits.
  The statement is: when the capital composition increases then the rate of profit decreases.
  Marx imposes that the rate of exploitation remains unchanged when the capital composition varies. For that reason, Robinson [10] considers this law as a tautology. Similarly, Sweezy [14] claims "The tendency of the rate of profit to fall follows directly from the assumption of a constant rate of surplus value and a rising organic composition of capital." For Mandel [5], "La loi n'est valable que toutes choses égales par ailleurs, c'est-à-dire à taux de plus-value constant."(the law is only valid if all other things are equal, i.e. the surplus rate is contant). For us, this criticism is too "harsch" as we will see below.

We construct a simple model in which we increase the technology. We obtain a decreasing rate of profitt accompanied by an augmentation of the capital composition and a lowering of the exploitation rate. This result shows that

- It is not necessary to maintain constant the rate of exploitation to have the law of falling rate of profit
- The criticism of Mandel [5] is wrong since "other things are not equal" (the exploitation rates changes)
- This law is not a tautology as Robinson [10] claimed since we can let the exploitation rate moving.
- Our result contradicts also the claim of Sweezy [14]
- Our result does not contradict Okishio [9] since if we diminish the costs by decreasing the technical input coefficients then the profit rate increases, because, in our simple model, the capital composition will decrease, the exploitation rate increases if we diminish the technical input coefficients.
- Our result is similar to the one of Romer [11]. The difference is that we show the capital composition increases when we improve the technology, while Romer did not show this point.

The paper is organized as follows.

- In Section 2 we present the economy.
- The values are defined in Section 3. We assume the technology matrix be productive in order to have strictly positive values.
- Sections 4 and 5 respectively deal with the rate of exploitation and the rate of profit.
- The Transformation problem is considered in Section 6
- Section 7 considers the law of the falling rate of profit
- The conclusion is in section 8
- An appendix on the properties of productive matrices is given at the end of this note.

2 The Economy

We consider a closed economy with 3 commodities. Commodity 1 and commodity 2 are capital goods (means of production). Commodity 3 is wage good. We assume

- To each industry there exists one and only one method of production
- Each industry produces one kind of input.
- There are no primary factors of production other than labour.

Producing the three goods requires capital goods and labour.
The technologies
To produce one unit of commodity 1 one needs a_{11} units of capital good 1, a_{21} units of capital good 2 and l_1 hours of labour.
To produce one unit of commodity 2 one needs a_{12} units of capital good 1, a_{22} units of capital good 2 and l_2 hours of labour.
To produce one unit of commodity 3 one needs a_{13} units of capital good 1, a_{23} units of capital good 2 and l_3 hours of labour.
The production process is composed of the lists of input of goods and labour in these amounts:

$$(a_{1i}, a_{2i}, l_i, i = 1, 2, 3)$$

3 The Values

Marx thought that values could be determined by technology alone, are not influenced by changes in wages and prices in the market, as long as the methods of production chosen remained unaffected. In the book "Capital" by Marx, there seem to be two definitions of value, which are:

- (a) 'All that these things now tell us is, that human labour-power has been expended in their production, that human labour is embodied in them. When looked at as crystals of this social substance, common to them all, they are -Values' (Capital, vol.1, page 38)
- (b) 'We see then that that which determines the magnitude of the value of any article is the amount of labour socially necessary, or the labour-time socially necessary for its production (Capital, vol.1, p.39)

In this note we will only consider the **values computed with definition (a)**
Let λ_1 denote the value of commodity 1 which is defined as the total amount of labour (in terms of labor-time) embodied (or materialized) in one unit of commodity 1. Define similarly the value λ_2 of commodity 2. The total labour

embodied in each commodity is

$$\lambda_1 = a_{11}\lambda_1 + a_{21}\lambda_2 + l_1 \tag{1}$$

$$\lambda_2 = a_{12}\lambda_1 + a_{22}\lambda_2 + l_2 \tag{2}$$

$$\lambda_3 = a_{13}\lambda_1 + a_{23}\lambda_2 + l_3 \tag{3}$$

Use matrices.

$$\begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} l_1 & l_2 \end{pmatrix} \tag{4}$$

$$\lambda_3 = \begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix} + l_3 \tag{5}$$

Define

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

We assume A is strictly positive, i.e., $a_{ij} > 0, \forall i, j$.

Remark 1 *Marx did not establish the positiveness of the values. He took that for granted.*

Positiveness of the values
Productiveness of the matrix A
The matrix A is said to be productive if there exists $x^0 \in \mathbb{R}^2_+ \setminus \{0\}$ such that $x^0 > Ax^0$.

Proposition 1 *The following statements are equivalent*

- *(i) The matrix A is productive*
- *(ii) There exists $x^0 > 0$, $x^0 > Ax^0$*
- *(iii) For any $f \geq 0$, there exists a non negative solution to the equation $x = Ax + f$*
- *(iv) The matrix $(I - A)^{-1}$ exists and has positive elements.*

Proof: See Appendix. ■

Assumption 1: A is productive.

Corollary 1 *Assume A productive and $a_{ij} > 0, \forall i, j$. Then the values $\lambda_1, \lambda_2, \lambda_3$ exist, are unique and strictly positive.*

4 Rate of exploitation

Let b denote the daily means of subsistence of the labourer. Let T be the prevailing length of the working day. The labour-time to obtain b is $\lambda_3 b$. The definition of rate of exploitation is

$$e = \frac{\textit{Unpaid labour}}{\textit{Paid labour}} = \frac{T - \lambda_3 b}{\lambda_3 b} = \frac{1 - \omega\lambda_3 b}{\omega\lambda_3 b}$$

where $\omega = \frac{1}{T}$.

We cite Marx: ' On the basis of capitalist production...this necessary labour $[\lambda_3 b]$ can form a part only of the working-day $[T]$ the working-day itself can never be reduced to this minimum (*Capital*, vol.1, p.232.) Thus Marx made the basic assumption of the theory of exploitation: $T > b\lambda_3$ (the exploitation rate e is strictly positive).

Assumption 2 The rate of exploitation is non negative: $e = \frac{1}{\omega\lambda_3 b} - 1 \geq 0$

Assumption 2 is weaker than the one Marx imposed.

5 The rates of profit

Lemma 1 *Suppose that no sector makes negative profit. In this case we have*

$$p_1/w \geq \lambda_1, p_2/w \geq \lambda_2$$

Proof: If no sector makes negative profit, we have

$$(p_1 \quad p_2) \geq (p_1 \quad p_2)A + w(l_1 \;\; l_2)$$

or equivalently

$$(p_1/w \quad p_2/w)(I - A) \geq (l_1 \;\; l_2)$$

Since the matrix A is productive, $(I - A)^{-1}$ is strictly positive. We then have

$$(p_1/w \;\; p_2/w) \geq (l_1 \;\; l_2)(I - A)^{-1}$$

Recall we also have

$$(\lambda_1 \;\; \lambda_2) = (l_1 \;\; l_2)(I - A)^{-1}$$

Hence $p_1/w \geq \lambda_1, p_2/w \geq \lambda_2$. ■

- Assume $Tw > p_3 b \Leftrightarrow 1 > \frac{p_3 b\omega}{w}$ (the worker can buy ωb amounts of wage-good with her/his hourly wages).
- Assume the rate of profit is uniform. Hence, $1 + \pi = \frac{p_3}{p_1 a_{13} + p_2 a_{23} + w l_3}$

We now state Marx main result.

Proposition 2 *Assume that no sector makes negative profit. Then* $e > \pi$

Proof: Recall

$$\lambda_3 = \lambda_1 a_{13} + \lambda_2 a_{23} + l_3$$

Therefore, using Lemma 1

$$\begin{aligned} 1+\pi &= \frac{p_3}{p_1 a_{13} + p_2 a_{23} + w l_3} \\ &\leq \frac{(p_3/w)}{\lambda_1 a_{13} + \lambda_2 a_{23} + l_3} = \frac{p_3}{w\lambda_3} < \frac{1}{\omega \lambda_3 b} = 1+e \end{aligned}$$

This implies $\pi < e$. ■

The following result is not mentioned in Morishima and for sure not in Marx.

Proposition 3 *Assume* $Tw \geq p_3 b \Leftrightarrow 1 \geq \frac{p_3 b \omega}{w}$*, and the profit rate is uniform. Then*

$$e = \pi \Leftrightarrow e = \pi = 0$$

In this case $Tw = p_3 b$*.*

Proof: If $e = \pi$ then

$$\begin{aligned} 1+\pi &= \frac{p_3}{p_1 a_{13} + p_2 a_{23} + w l_3} \\ &= \frac{(p_3/w)}{\lambda_1 a_{13} + \lambda_2 a_{23} + l_3} = \frac{p_3}{w\lambda_3} = \frac{1}{\omega \lambda_3 b} = 1+e \end{aligned}$$

We claim that $p_1/w = \lambda_1, p_2/w = \lambda_2$. Indeed, suppose $p_1/w > \lambda_1$. In this case we have a contradiction

$$\begin{aligned} 1+\pi &= \frac{p_3}{p_1 a_{13} + p_2 a_{23} + w l_3} \\ &< \frac{(p_3/w)}{\lambda_1 a_{13} + \lambda_2 a_{23} + l_3} = \frac{p_3}{w\lambda_3} \leq \frac{1}{\omega \lambda_3 b} = 1+e \end{aligned}$$

This implies $\pi < e$.
Since

$$\begin{pmatrix} p_1/w & p_2/w \end{pmatrix} = (1+\pi)(\begin{pmatrix} p_1/w & p_2/w \end{pmatrix} A + \begin{pmatrix} l_1 & l_2 \end{pmatrix})$$

and $p_1/w = \lambda_1, p_2/w = \lambda_2$ we obtain

$$\begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} = (1+\pi)(\begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} A + \begin{pmatrix} l_1 & l_2 \end{pmatrix})$$

But the values determination equation gives

$$\begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} A + (l_1 \;\; l_2)$$

Gather the two previous equations:

$$\begin{pmatrix}\lambda_1 & \lambda_2\end{pmatrix} = (1+\pi)\begin{pmatrix}\lambda_1 & \lambda_2\end{pmatrix}$$

This implies $\pi = 0$. Since we assume $e = \pi$, hence $e = 0$. But $e = \pi$ implies $\frac{p_3}{w\lambda_3} = \frac{1}{\omega\lambda_3 b}$ or $wT = p_3 b$. ■

This result cannot be found in Marx since he assumed $e > 0$.

6 The static transformation of values in production prices

Let (p_1, p_2, p_3) denote the prices of capital goods and of the wage good. It is assumed that wages are fixed at a level at which workers can only purchase their daily means of subsistence. Hence $w = p_3\omega b$, To define profits we will first define the compositions of the capital.

Constant capitals

For sector $i = 1, 2, 3,\ C_i = \lambda_1 a_{1i} + \lambda_2 a_{2i}$

Variable capitals

For sector $i = 1, 2, 3,\ V_i = \omega\lambda_3 b l_i$

Constant capitals in terms of prices

For sector $i = 1, 2, 3,\ C_i^p = p_1 a_{1i} + p_2 a_{2i}$

Variable capitals in terms of prices

For sector $i = 1, 2, 3,\ V_i^p = \omega p_3 b l_i$

where $\omega = \frac{1}{T}$.
The profits of each sector are

$$\begin{aligned}\text{For sector } i = 1, 2, 3, \Pi_i &= p_i - p_1 a_{1i} - p_2 a_{2i} - \omega p_3 b l_i \\ &= p_i - (C_i^p + V_i^p)\end{aligned}$$

while the surplus are

$$\begin{aligned}\text{For sector } i = 1, 2, 3, S_i &= \lambda_i - \lambda_1 a_{1i} - \lambda_2 a_{2i} - \omega\lambda_3 b l_i \\ &= \lambda_i - (C_i + V_i)\end{aligned}$$

The rate of profit in sector i is thus

$$\pi_i = \frac{\Pi_i}{C_i^p + V_i^p}$$

Let (p_1, p_2, p_3) be a system of prices which are strictly positive. We say that they are *production prices* if, using them to calculate the profit rates of the sectors, these profit rates are equal. Denote this common value by π. We then have

$$\forall i, \ \ \pi = \frac{\Pi_i}{C_i^p + V_i^p}$$

We define *undiscounted production prices* as a system of strictly positive production prices (q_1, q_2, q_3) which satisfy $q_1 + q_2 + q_3 = 1$.

Lemma 2 *There exists a unique uniform profit rate and a unique system of undiscounted production prices.*

Proof: Let $(p_1, p_2, p_3) >> 0$ be a system of production prices associated with a uniform rate of profit π. They satisfy

$$p_i = (1 + \pi)(p_1 a_{1i} + p_2 a_{2i} + \omega b p_3 l_i)$$

for any $i = 1, 2, 3$. Consider the matrix

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \omega b l_1 & \omega b l_2 & \omega b l_3 \end{pmatrix}$$

The matrix M is called the matrix of augmented input coefficient matrix (see Roemer [11]).
In a compact form

$$\begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix} = (1 + r) \begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix} M$$

Since M is strictly positive, $\frac{1}{1+r}$ is the largest single left eigenvalue of M and (p_1, p_2, p_3) is a left associated positive eigenvector. This one is defined up to a scalar. Choose (p_1, p_2, p_3) such that $p_1 + p_2 + p_3 = 1$. Such an eigenvector is unique. ■

We say that there is a *transformation of values in prices* if there exists a price system (p_1, p_2, p_3) and a positive number α such that $\lambda_i = \alpha p_i, \forall i = 1, 2, 3.$.

Lemma 3 *Let*

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \omega b l_1 & \omega b l_2 & \omega b l_3 \end{pmatrix}$$

M *is productive.*

Proof: Recall

$$\begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} l_1 & l_2 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \omega\lambda_3 b \begin{pmatrix} l_1 & l_2 \end{pmatrix} + e\omega\lambda_3 b \begin{pmatrix} l_1 & l_2 \end{pmatrix}$$

$$> \begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \omega\lambda_3 b \begin{pmatrix} l_1 & l_2 \end{pmatrix}$$

And

$$\lambda_3 = \begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix} + l_3$$

$$\lambda_3 = \begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix} + \omega\lambda_3 b l_3 + e\omega\lambda_3 b$$

$$\lambda_3 > \begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix} + \omega\lambda_3 b l_3$$

In a compact form

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} > \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} M$$

This means M is productive (see Appendix). ∎

6.1 Endogenous Necessary and sufficient conditions to transform values into production prices

The proof of the following lemma can be found also Morishima "Marx's Economics" (Cambridge University Press, 1973).

Lemma 4 *Let (p_1, p_2, p_3) be a system of undiscounted production prices. It is unique, from Lemma 2. Calculate the profits (Π_1, Π_2, Π_3) with these prices. We have*

$$\frac{\Pi_1}{S_1} = \frac{\Pi_2}{S_2} = \frac{\Pi_3}{S_3} \Leftrightarrow \exists \alpha > 0, \ \ \lambda_i = \alpha p_i, \ \forall i$$

Proof: (a) Assume $\frac{\Pi_1}{S_1} = \frac{\Pi_2}{S_2} = \frac{\Pi_3}{S_3} = \frac{1}{\alpha} > 0$. Since, $\forall i$

$$S_i = \lambda_i - (C_i + V_i)$$
$$\Pi_i = p_i - (C_i^p + V_i^p)$$

we have

$$\begin{pmatrix} \lambda_1 - \alpha p_1 & \lambda_2 - \alpha p_2 & \lambda_3 - \alpha p_3 \end{pmatrix} = \begin{pmatrix} \lambda_1 - \alpha p_1 & \lambda_2 - \alpha p_2 & \lambda_3 - \alpha p_3 \end{pmatrix} M$$

with

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \omega b l_1 & \omega b l_2 & \omega b l_3 \end{pmatrix}$$

Equivalently

$$\begin{pmatrix} \lambda_1 - \alpha p_1 & \lambda_2 - \alpha p_2 & \lambda_3 - \alpha p_3 \end{pmatrix} (I - M) = 0$$

From Lemma 3, M is productive implying $(I - M)$ is invertible. Hence $\lambda_i = \alpha p_i, \ \forall i$.
(b) Conversely, assume $\lambda_i = \alpha p_i, \forall i$ with some $\alpha > 0$. Using the expressions

$$S_i = \lambda_i - (C_i + V_i)$$
$$\Pi_i = p_i - (C_i^p + V_i^p)$$

we easily obtain $S_i = \alpha \Pi_i, \ \forall i$. ■

Lemma 5 *(a) If* $\begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix}$ *is a system of production prices, then*

$$\frac{S_1}{\Pi_1} = \frac{S_2}{\Pi_2} = \frac{S_3}{\Pi_3} \Rightarrow \frac{C_1}{V_1} = \frac{C_2}{V_2} = \frac{C_3}{V_3}$$

(b) Conversely, assume $\frac{C_1}{V_1} = \frac{C_2}{V_2} = \frac{C_3}{V_3}$. *Then there exists a uniform rate of profits* π *and a system of associated production prices (defined up to a positive scalar) such that* $\frac{S_1}{\Pi_1} = \frac{S_2}{\Pi_2} = \frac{S_3}{\Pi_3}$.

The following proof is adapted from the one in Morishima "Marx's Economics" (Cambridge University Press, 1973).
Proof: (a) Assume $\frac{S_1}{\Pi_1} = \frac{S_2}{\Pi_2} = \frac{S_3}{\Pi_3}$. From Lemma 4, there exists $\alpha > 0$ s.t. $\lambda_i = \alpha p_i, \ \forall i$. Let π denote the associated uniform rate of profit. We have

$$\forall i, \ \ \pi = \frac{\Pi_i}{C_i^p + V_i^p} = \frac{S_i}{C_i + V_i} = \frac{eV_i}{C_i + V_i} = \frac{e}{C_i/V_i + 1}$$

Hence $\frac{C_i}{V_i} = \frac{e}{\pi} - 1$, for all i.
(b) Conversely, assume $\frac{C_1}{V_1} = \frac{C_2}{V_2} = \frac{C_3}{V_3}$. Consider the ratios $\frac{eV_i}{C_i + V_i}$ which are the same for all i. Denote this common value by π. That means $\pi = \frac{eV_i}{C_i + V_i}$. We have

$$\lambda_i = (1 + \pi)(C_i + V_i)$$

Take some $\alpha > 0$. Define the prices (p_1, p_2, p_3) by $p_i = \frac{\lambda_i}{\alpha}$. Since

$$\begin{aligned} S_i &= \lambda_i - (C_i + V_i) \\ &= \pi(C_i + V_i) \\ &= \alpha\pi(C_i^p + V_i^p) \\ &= \alpha \Pi_i \end{aligned}$$

We have $\frac{S_i}{\Pi_i} = \alpha, \forall i$.
To end the proof we will show that the prices (p_1, p_2, p_2) are defined up to a positive scalar. Since

$$\lambda_i = (1+\pi)(C_i + V_i)$$

we have

$$p_i = (1+\pi)(p_1 a_{1i} + p_2 a_{2i} + \omega b p_3 l_i)$$

In a compact form

$$\begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix} = (1+\pi) \begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix} M$$

Since M is strictly positive, $\frac{1}{1+\pi}$ is the largest left eigenvalue of M and $\begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix}$ is the associated positive eigenvector (defined up to a positive scalar). ■

Proposition 4

$$\{\frac{C_1}{V_1} = \frac{C_2}{V_2} = \frac{C_3}{V_3}\} \Leftrightarrow$$

$\{\exists \alpha > 0, \exists$ *undiscounted production prices* (p_1, p_2, p_3) *such that* $\lambda_i = \alpha p_i,\ \forall i = 1, 2, 3\}$

Proof: (a) Assume $\frac{C_1}{V_1} = \frac{C_2}{V_2} = \frac{C_3}{V_3}$. Let e denote the rate of exploitation. Define π

$$\pi = \frac{eV_1}{C_1 + V_1} = \frac{eV_2}{C_2 + V_2} = \frac{eV_3}{C_3 + V_3}$$

We then have

$$\lambda_i = (1+\pi)(C_i + V_i),\ \forall i = 1, 2, 3.$$

Define $p_i = \frac{\lambda_i}{\alpha},\ \forall i = 1, 2, 3$ and $\alpha > 0$ is chosen such that $p_1 + p_2 + p_3 = 1$. In other words (p_i) is a system of undiscounted prices. We observe we then have

$$\forall i,\ p_i = (1+\pi)(p_1 a_{1i} + p_2 a_{2i} + p_3 \omega b l_3) = (1+\pi)(C_i^p + V_i^p)$$

The price system (p_1, p_2, p_3) is a system of undiscounted production prices.
(b) Conversely, assume $\lambda_i = \alpha p_i, \forall i = 1, 2, 3$ with $\alpha > 0$ and (p_1, p_2, p_3) is an undiscounted production prices system. Recall that $\forall i, \lambda_i = C_i + V_i + S_i = C_i + (1+e)V_i$. Let π be the uniform rate of profit associated with the prices (p_1, p_2, p_3). We have

$$\forall i,\ p_i = (1+\pi)(p_1 a_{1i} + p_2 a_{2i} + p_3 \omega b l_3)$$

or

$$\frac{\lambda_i}{\alpha} = (1+\pi)\frac{1}{\alpha}(\lambda_1 a_{1i} + \lambda_2 a_{2i} + \lambda_3 \omega b l_3) = (1+\pi)\frac{1}{\alpha}(C_i + V_i)$$

thus

$$C_i + (1+e)V_i = (1+\pi)(C_i + V_i)$$

and

$$\frac{C_i}{V_i} = \frac{e}{\pi} - 1, \forall i \tag{6}$$

■

Proposition 4 gives endogenous necessary and sufficient conditions for the transformation of values in production prices. These conditions involve prices and values. Actually we can have exogenous necessary and sufficient conditions for this problem which involve only the means of production and used labour-times.

6.2 Exogenous necessary and sufficient for the transformation of values into prices

Recall

$$\frac{C_1}{V_1} = \frac{\lambda_1 a_{11} + \lambda_2 a_{21}}{\lambda_3 \omega b l_1} = \frac{(\lambda_1 \;\; \lambda_2)\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}}{\lambda_3 \omega b l_1} \tag{7}$$

$$\frac{C_2}{V_2} = \frac{\lambda_1 a_{12} + \lambda_2 a_{22}}{\lambda_3 \omega b l_2} = \frac{(\lambda_1 \;\; \lambda_2)\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}}{\lambda_3 \omega b l_2} \tag{8}$$

$$\frac{C_3}{V_3} = \frac{\lambda_1 a_{13} + \lambda_2 a_{23}}{\lambda_3 \omega b l_3} = \frac{(\lambda_1 \;\; \lambda_2)\begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix}}{\lambda_3 \omega b l_3} \tag{9}$$

But from the determination equations of the values (λ_1, λ_2):

$$\begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} = \begin{pmatrix} l_1 & l_2 \end{pmatrix} (I - A)^{-1}$$

We then get

$$\frac{C_1}{V_1} = \frac{\begin{pmatrix} l_1 & l_2 \end{pmatrix} (I - A)^{-1} \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}}{\lambda_3 \omega b l_1}$$

$$\frac{C_2}{V_2} = \frac{\begin{pmatrix} l_1 & l_2 \end{pmatrix} (I - A)^{-1} \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}}{\lambda_3 \omega b l_2}$$

$$\frac{C_3}{V_3} = \frac{\begin{pmatrix} l_1 & l_2 \end{pmatrix} (I - A)^{-1} \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix}}{\lambda_3 \omega b l_3}$$

And we have $\{\frac{C_1}{V_1} = \frac{C_2}{V_2} = \frac{C_3}{V_5}\}$ if and only if

$$\frac{\begin{pmatrix} l_1 & l_2 \end{pmatrix} (I - A)^{-1} \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}}{l_1} = \frac{\begin{pmatrix} l_1 & l_2 \end{pmatrix} (I - A)^{-1} \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}}{l_2} = \frac{\begin{pmatrix} l_1 & l_2 \end{pmatrix} (I - A)^{-1} \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix}}{l_3}$$

Condition $\mathcal{C}$

$$\frac{\begin{pmatrix} l_1 & l_2 \end{pmatrix}(I-A)^{-1}\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}}{l_1} = \frac{\begin{pmatrix} l_1 & l_2 \end{pmatrix}(I-A)^{-1}\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}}{l_2} = \frac{\begin{pmatrix} l_1 & l_2 \end{pmatrix}(I-A)^{-1}\begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix}}{l_3}$$

We state a corollary of Proposition 4

Corollary 2 *There exists a unique transformation of values in production prices with the form $\forall i, \lambda_i = \alpha p_i, \alpha > 0$ if, and only if, Condition $\mathcal{C}$ holds.*

7 The Law of the Falling rate of Profit

In this section, the profit rate is uniform, the compositions of capital are equal through the sectors.
We quote Marx: *"The gradual growth of constant capital in relation to variable capital must necessaily lead to a gradual fall of the general rate of profit, so long as the rate of surplus-value, or the intensity of exploitation of labour by capital, remain the same"* (*Capital*, III, p.212).
An important assumption in this statement is that the exploitation rate e remains constant.
Consider the relation (6). Since $\frac{C_i}{V_i} = \frac{C}{V}$ with $C = C_1+C_2+C_3,\ V = V_1+V_2+V_3$ we have

$$\pi = \frac{e}{1 + C/V} \tag{10}$$

Therefore we can state : *Suppose the exploitation rate e remains unchanged. Then, if C/V increases then π decreases.*
The law of the falling rate of profit is satisfied if e can be always kept constant.
Consider relations (7), (8),(9),

$$\frac{C_1}{V_1} = \frac{\lambda_1 a_{12} + \lambda_2 a_{22}}{\lambda_3 \omega b l_2} = \frac{\begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix}\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}}{\lambda_3 \omega b l_1} = \frac{\begin{pmatrix} l_1 & l_2 \end{pmatrix}(I-A)^{-1}\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}}{\lambda_3 \omega b l_2}$$

$$\frac{C_2}{V_2} = \frac{\lambda_1 a_{12} + \lambda_2 a_{22}}{\lambda_3 \omega b l_2} = \frac{\begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix}\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}}{\lambda_3 \omega b l_2} = \frac{\begin{pmatrix} l_1 & l_2 \end{pmatrix}(I-A)^{-1}\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}}{\lambda_3 \omega b l_2}$$

$$\frac{C_3}{V_3} = \frac{\lambda_1 a_{13} + \lambda_2 a_{23}}{\lambda_3 \omega b l_3} = \frac{\begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix}\begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix}}{\lambda_3 \omega b l_3} = \frac{\begin{pmatrix} l_1 & l_2 \end{pmatrix}(I-A)^{-1}\begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix}}{\lambda_3 \omega b l_3}$$

Recall $\frac{C_1}{V_1} = \frac{C_2}{V_2} = \frac{C_3}{V_3} = \frac{C}{V}$. Recall also $(1+e)(\omega\lambda_3 b) = 1$. Now assume $a_{ij} = a > 0, \forall i = 1,2,\ \forall j = 1,2,3$, and $2a < 1$. In this case the matrix A is productive. Tedious calculations give

$$(I-A)^{-1} = \frac{1}{1-2a}\begin{pmatrix} 1-a & a \\ a & 1-a \end{pmatrix}$$

The expressions of C/V are:

$$\frac{C}{V} = \frac{C_1}{V_1} = (1+e)\frac{\begin{pmatrix} l_1 & l_2 \end{pmatrix}(I-A)^{-1}\begin{pmatrix} a \\ a \end{pmatrix}}{l_1}$$

$$\frac{C}{V} = \frac{C_2}{V_2} = (1+e)\frac{\begin{pmatrix} l_1 & l_2 \end{pmatrix}(I-A)^{-1}\begin{pmatrix} a \\ a \end{pmatrix}}{l_2}$$

$$\frac{C}{V} = \frac{C_3}{V_3} = (1+e)\frac{\begin{pmatrix} l_1 & l_2 \end{pmatrix}(I-A)^{-1}\begin{pmatrix} a \\ a \end{pmatrix}}{l_3}$$

Condition $\mathcal{C}$ becomes $l_1 = l_2 = l_3 = l$ (the labour times are the same in all the sectors). We easily find $\frac{C}{V} = (1+e)\frac{2a}{1-2a}$. If e is unchanged, to increase the capital composition C/V we have to increase a and can let l unchanged. However, since $l_1 = l_2 = l$ and

$$(\lambda_1 \ \ \lambda_2) = (l_1 \ \ l_2)(I-A)^{-1}$$

we obtain

$$\lambda_1 = \lambda_2 = \frac{l}{1-2a}$$

Now since $\lambda_3 = \lambda_1 a + \lambda_2 a + l_3$ we find $\lambda_3 = \frac{l}{1-2a}$. It increases when a increases implying e decreases. That contradictis the assumption that e remains unchanged.

However, strengthen the condition on the technology in order to have a positive exploitation rate. Indeed, since $(1+e) = \frac{1}{\omega b \lambda_3} = \frac{1-2a}{\omega b} \Rightarrow e = \frac{1-\omega b-2a}{\omega b}$. Hence $e > 0 \Leftrightarrow 2a < 1-\omega b$. Assume $2a < 1-\omega b$.

We then have

$$\frac{C}{V} = \frac{2a}{\omega b}, \ \ \pi = \frac{1}{\omega b + 2a} - 1$$

and

$$\frac{C}{V} \uparrow \Leftrightarrow a \uparrow \Leftrightarrow \pi \downarrow$$

We can state

Proposition 5 *Assume furthermore $2a < 1-\omega b$. In this case, if $\frac{C}{V}$ increases then the rate of profit π decreases, while the exploitation rate e decreases.*

Summing up, under a stronger assumption on the the technology and on the subsistence level b or the lenght of the work duration T (b is small or/and T is large), the law of falling rate of profit holds without imposing the exploitation rate to be unchanged.

8 Conclusion

We summarize our contributions. With a very simple model in the line of Morishima [8] we obtain

- We give exogenous conditions to transform the values into production prices
- The law of the falling rate of profit does not require the exploitation rate be constant. This contradicts the criticisms of Robinson, Sweezy, Mandel. Our result does not contradict Romer and Okishio.

Appendix

Let $A = (a_{ij}), i = 1, \ldots, n; j = 1, \ldots, n$ be a $n \times n$ matrix which satisfies $a_{ij} > 0, \forall i, j$.
We say that A is productive if there exists $p \in \mathbb{R}^n_{++}$ such that $p > Ap$ in the sense the vector $p - Ap$ has strictly positive components.
The proof of the following lemma is borrowed from Wikipedia.

Lemma 6 *A is productive iff the matrix $(I - A)^{-1}$ exists and has no negative coefficients.*

Proof: (i) We prove: if $(I - A)^{-1}$ exists and has no negative coefficients, then A is productive.
Indeed, take $q \in \mathbb{R}^n_{++}$. Let $p = (I - A)^{-1}q$. Then $p >> 0$. We have $(I - A)p = q > 0$. Thus A is productive.
(ii) Now suppose A productive and $(I - A)^{-1}$ does not exist. Let $p \in \mathbb{R}^n_{++}$ satisfy $p > Ap$. Define $v = p - Ap$. Then $v_k > 0, \forall k$. Let $Z = \ker(I - A)$. There must exist $z \in Z$ with $z_i > 0$.
Define $c = \sup_i \frac{z_i}{p_i}$. We can find k such that $c = \frac{z_k}{p_k}$. Then

$$0 < cv_k = cp_k - \sum_{i=1}^{n} a_{ki}cp_i = z_k - \sum_{i=1}^{n} a_{ki}cp_i \leq z_k - \sum_{i=1}^{n} a_{ki}z_i = 0$$

We have a contradiction. Hence $I - A$ is invertible.
We now prove that the coefficients of $(I - A)^{-1}$ are non negative. Suppose the contrary. This matrix has a negative coefficient. In his case there exists a vector $x \geq 0$ such that the vector $y = (I - A)^{-1}x$ has at least a negative component.
Define

$$c = \sup_i \frac{-y_i}{p_i}$$

Suppose $c = \frac{-y_k}{p_k}$. We have

$$0 < cv_k = c(p_k - \sum_{i=1}^{n} a_{ki}p_i) \leq -y_k + \sum_{i=1}^{n} a_{ki}y_i = -x_k \leq 0$$

We have a contradiction. Hence, the coefficients of $(I - A)^{-1}$ are non negative.
■

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