A simple mathematical model of Marx's Economics

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This note is based on the chapters in Part I and Part II of the book "Marx's Economics" (Cambridge University Press, 1973) by Michio Morishima.
The quotations come from Karl Marx's *Capital*, Progress Publishers, Moscow (vol.1, 1965).
This note does not cover all the problems raised in Marx economics, such as, the transformation, reproduction...It must be viewed as an introduction to Marx's economics by using mathematics.

Part I: The Labour Theory of Value

Chapter 1: Dual definition of value

We consider a closed economy with 3 commodities. Commodity 1 and commodity 2 are capital goods (means of production). Commodity 3 is wage good. We assume

- To each industry there exists one and only one method of production
- Each industry produces one kind of input.
- There are no primary factors of production other than labour.

Producing the three goods requires capital goods and labour.
Marx thought that values could be determined by technology alone, are not influenced by changes in wages and prices in the market, as long as the methods of production chosen remained unaffected. In the book "Capital" by Marx, there seem to be two definitions of value, which are:

- (a) 'All that these things now tell us is, that human labour-power has been expended in their production, that human labour is embodied in them. When looked at as crystals of this social substance, common to them all, they are -Values' (Capital, vol.1, page 38)
- (b) 'We see then that that which determines the magnitude of the value of any article is the amount of labour socially necessary, or the labour-time socially necessary for its production (Capital, vol.1, p.39)

The aim of this chapter is to give a rigorous proof that the values computed from these two definitions coincide under the "hidden" assumptions which will be exhibited in the next chapter.
The technologies
To produce one unit of commodity 1 one needs a_{11} units of capital good 1, a_{21} units of capital good 2 and l_1 hours of labour.
To produce one unit of commodity 2 one needs a_{12} units of capital good 1, a_{22} units of capital good 2 and l_2 hours of labour.
To produce one unit of commodity 3 one needs a_{13} units of capital good 1, a_{23} units of capital good 2 and l_3 hours of labour.
The production process is composed of the lists of input of goods and labour in these amounts:

$$(a_{1i}, a_{2i}, l_i, i = 1, 2, 3)[\text{ this list is exogenously given }]$$

Values computed with definition (a)

Let λ_1 denote the value of commodity 1 which is defined as the total amount of labour (in terms of labor-time) embodied (or materialized) in one unit of commodity 1. Define similarly the value λ_2 of commodity 2. The total labour embodied in each commodity is

$$\lambda_1 = a_{11}\lambda_1 + a_{21}\lambda_2 + l_1 \tag{1}$$

$$\lambda_2 = a_{12}\lambda_1 + a_{22}\lambda_2 + l_2 \tag{2}$$

$$\lambda_3 = a_{13}\lambda_1 + a_{23}\lambda_2 + l_3 \tag{3}$$

Use matrices.

$$\begin{pmatrix}\lambda_1 & \lambda_2\end{pmatrix} = \begin{pmatrix}\lambda_1 & \lambda_2\end{pmatrix}\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix} + \begin{pmatrix}l_1 & l_2\end{pmatrix} \tag{4}$$

$$\lambda_3 = \begin{pmatrix}\lambda_1 & \lambda_2\end{pmatrix}\begin{pmatrix}a_{13}\\ a_{23}\end{pmatrix} + l_3 \tag{5}$$

Values computed with definition (b)

The value of a commodity is the total amount of labour to produce a unit of that commodity with the method of production prevailing in the society.

For instance, in order to have one more unit of capital good 1, a_{11} units of capital good 1, a_{21} units of capital good 2 are required. Suppose the final production of good 1 is q_1^1. This production requires a consumption $a_{11}q_1^1$ of good 1, and a consumption in sector 2, $a_{12}q_2^1$ of good 1.

We suppose there is no need to have more of good 2. But this good must be produced since it it is necessary for the production of good 1. The quantity of good 2 equals the consumptions of capital good 2, $a_{21}q_1^1$ and $a_{22}q_2^1$.

We then have to solve the system

$$q_1^1 = a_{11}q_1^1 + a_{12}q_2^1 + 1$$
$$q_2^1 = a_{21}q_1^1 + a_{22}q_2^1$$

We can write that the net productions equal the demands

$$q_1^1 - (a_{11}q_1^1 + a_{12}q_2^1) = 1$$
$$q_2^1 - (a_{21}q_1^1 + a_{22}q_2^1) = 0$$

Similarly, to have only one more unit of good 2, we solve the system

$$q_1^2 = a_{11}q_1^2 + a_{12}q_2^2$$
$$q_2^2 = a_{21}q_1^2 + a_{22}q_2^2 + 1$$

In terms of marices

$$\begin{pmatrix} q_1^1 \\ q_2^1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} q_1^1 \\ q_2^1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{6}$$

$$\begin{pmatrix} q_1^2 \\ q_2^2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} q_1^2 \\ q2_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{7}$$

The total amount of labour necessary to produce one unit of capital good 1 is

$$\mu_1 = l_1 q_1^1 + l_2 q_2^1$$

And for capital good 2

$$\mu_2 = l_1 q_1^2 + l_2 q_2^2$$

Proposition 1 *We have* $\lambda_1 = \mu_1,\ \lambda_2 = \mu_2$

Proof: Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ Denote $Q = \begin{pmatrix} q_1^1 & q_1^2 \\ q_2^1 & q_2^2 \end{pmatrix}$ One can check that

$$Q = AQ + I$$

where I the identity-matrix.
Consider relation (4). Postmultiply this relation by Q. We get

$$(\lambda_1 \ \ \lambda_2)Q = (\lambda_1 \ \ \lambda_2)AQ + (l_1 \ \ l_2)Q$$

Equivalently

$$(\lambda_1 \ \ \lambda_2)(Q - AQ) = (l_1 \ \ l_2)Q$$

Since $Q - AQ = I$, we have

$$(\lambda_1 \ \ \lambda_2) = (l_1 \ \ l_2)Q = (\mu_1 \ \ \mu_2)$$

We have proved $\lambda_1 = \mu_1,\ \lambda_2 = \mu_2$. ■

In sector 3 (wage) it is required to have the quantities a_{13}, a_{23} of capital goods 1 and 2. These quantities are obtained by producing in this sector 3, the quantities x_1^3, x_2^3 of capital goods. They equal the consumption of capital goods 1 and 2 and the final demands a_{13}, a_{23}. Explicitly they solve the system

$$x_1^3 = a_{11}x_1^3 + a_{12}x_2^3 + a_{13}$$
$$x_2^3 = a_{21}x_1^3 + a_{22}x_2^3 + a_{23}$$

In terms of matrices

$$\begin{pmatrix} x_1^3 \\ x_2^3 \end{pmatrix} = A \begin{pmatrix} x_1^3 \\ x_2^3 \end{pmatrix} + \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix}$$

The total amount of labor to produce a_{13}, a_{23} is

$$\mu_3 = l_1 x_1^3 + l_2 x_2^3 + l_3$$

From (5) the value of wage good is

$$\lambda_3 = a_{13}\lambda_1 + a_{23}\lambda_2 + l_3$$

Proposition 2 $\mu_3 = \lambda_3$

Proof: Consider again relation (4). Postmultiply it by $\begin{pmatrix} x_1^3 \\ x_2^3 \end{pmatrix}$ We get

$$(\lambda_1 \ \ \lambda_2) \begin{pmatrix} x_1^3 \\ x_2^3 \end{pmatrix} = (\lambda_1 \ \ \lambda_2) A \begin{pmatrix} x_1^3 \\ x_2^3 \end{pmatrix} + (l_1 \ \ l_2) \begin{pmatrix} x_1^3 \\ x_2^3 \end{pmatrix}$$

Equivalently

$$(\lambda_1 \ \ \lambda_2)(\begin{pmatrix} x_1^3 \\ x_2^3 \end{pmatrix} - A \begin{pmatrix} x_1^3 \\ x_2^3 \end{pmatrix}) = (l_1 \ \ l_2) \begin{pmatrix} x_1^3 \ x_2^3 \end{pmatrix}$$

i.e.

$$(\lambda_1 \ \ \lambda_2) \begin{pmatrix} a_3^1 \\ a_3^2 \end{pmatrix} = (l_1 \ \ l_2) \begin{pmatrix} x_1^3 \ x_2^3 \end{pmatrix}$$

or

$$\lambda_3 - l_3 = \mu_3 - l_3 \Leftrightarrow \lambda_3 = \mu_3.$$

■

Remark 1 *Marx did not establish the positiveness of the values. He took that for granted.*
Moreover, there are hidden assumptions which are:
Assumption H1 *for any* $(l_1 \ \ l_2)$ *the system (4)*

$$\begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} l_1 & l_2 \end{pmatrix}$$

has a solution.
Assumption H2 *for any* $(f_1 \ \ f_2) > 0$ *the system*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

has a solution.

Positiveness of the values

Productiveness of the matrix A

The matrix A is said to be productive if there exists $x^0 \in \mathbb{R}^2_+ \setminus \{0\}$ such that $x^0 > Ax^0$.

Proposition 3 *The following statements are equivalent*

- *(i) The matrix A is productive*
- *(ii) There exists $x^0 > 0$, $x^0 > Ax^0$*
- *(iii) For any $f \geq 0$, there exists a non negative solution to the equation $x = Ax + f$*

Proof: (i)$\Leftrightarrow$(ii): from the very definition of productiveness.
(iii)$\Rightarrow$ (ii): take $f > 0$. Then there exists a solution x^0 to the equation $x = Ax + f$. Obviously $x^0 > Ax^0$.
(ii)$\Rightarrow$ (iii): Let $x^0 > 0$ satisfy $x^0 > Ax^0$. Take any $f \geq 0$. There exists $t > 0$ sufficiently large, for which we have $tx^0 > Atx^0 + f$. Let $x^1 = Atx^0 + f$. Then $tx^0 > x^1$. This implies $x^1 \geq Ax^1 + f$. Define $x^2 = Ax^1 + f$. Then $x^1 \geq x^2$ and $x^2 \geq Ax^1 + f \geq Ax^2 + f$. Define $x^3 = Ax^2 + f$ and so on.
We construct a decreasing sequence $(tx^0, x^1, x^2, \dots,)$ bounded below by f.Thus $x^n \to x \geq 0$ which satisfies $x = Ax + f$. ■

Lemma 1 *Assume $H1$. Then if A is productive then the transposed ${}^tA$ is productive.*
Assume $H2$. Then if ${}^tA$ is productive then A is productive.
Consequently, if we assume $H1$ and $H2$ then A is productive iff ${}^tA$ is productive.

Proof: Let A be productive. Let $g = (g_1 \quad g_2) > 0$. Then the equation $y = yA + g$ has a solution (because of H1). Take $f > 0$. Then there exists $x > 0$ solution to $x = Ax + f$. This leads to

$$yx = yAx + yf$$

and

$$yx = yAx + gx$$

Hence

$$yf = gx > 0$$

Take successively $f = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,$f = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
We have $y > 0$ and $y > yA \Leftrightarrow^t y >^t A^t y$. That means ${}^tA$ is productive.
We use the same kind of proof to have the converse, when we assume $H2$. ■

Proposition 4 *Assume H1 and H2. Assume also A productive and* $a_{ij} > 0, \forall i, j$. *Assume* $a_{13} > 0, a_{23} > 0, l_1 > 0, l_2 > 0, l_3 > 0$. *Then the values* $\lambda_1, \lambda_2, \lambda_3,\ \mu_1, \mu_2, \mu_3$ *exist and are strictly positive.*

Proof: Since $(\lambda_1\ \ \lambda_2)$ is a non negative solution to (4), and since $(l_1, l_2) >> 0$ they are actually strictly positive. From (5), $\lambda_3 > 0$.
Under H1 and H2, (μ_1, μ_2, μ_3) exist and equal $(\lambda_1, \lambda_2, \lambda_3)$. ■

Remark 2 *We will prove that the values are actually unique when we have only two capital goods.*

Lemma 2 *Assume* $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ *productive. Assume also* $a_{ij} > 0, \forall i, j$. *Then* $I - A$ *is invertible.*

Proof: Let x^0 satisfy $x^0 > Ax^0$ i.e.

$$x_1^0 > a_{11}x_1^0 + a_{12}x_2^0$$
$$x_2^0 > a_{21}x_1^0 + a_{22}x_2^0$$

This leads to

$$(1 - a_{11}) > 0,\ (1 - a_{22}) > 0,\ x_1^0[(1 - a_{11})(1 - a_{22}) - a_{12}a_{21}] > 0$$

But $\det(I - A) = [(1 - a_{11})(1 - a_{22}) - a_{12}a_{21}] > 0$. ■

A general proof is given in the appendix.

Lemma 3 *If A is productive, then for any* $f > 0$ *there exists a unique positive solution to the equation* $x = Ax + f$.

Proof: Since A is productive, there exists a positive solution to the equation $x = Ax + f$. Since $I - A$ is invertible, this solution is unique. ■

Proposition 5 *Assume A productive and* $a_{ij} > 0, \forall i, j$. *Assume* $a_{13} > 0, a_{23} > 0, l_1 > 0, l_2 > 0, l_3 > 0$. *Then the values* $\lambda_1, \lambda_2, \lambda_3,\ \mu_1, \mu_2, \mu_3$ *exist, are unique and strictly positive.*

We say that a society is capitalist if the technology $(a_{ij}), i = 1, 2, 3;\ j = 1, 2$ prevails in this society.

This society is said to be viable if there exists a system of prices and wage $(p_1, p_2, p_3, w) > 0$ which satisfy

$$p_1 > p_1 a_{11} + p_2 a_{21} + w l_1$$
$$p_2 > p_1 a_{12} + p_2 a_{22} + w l_2$$
$$p_3 > p_1 a_{13} + p_2 a_{13} + w l_3$$

That means using this system of prices and wage to evaluate, the cost of a production process is less than the prices of its output.
But observe

$$p_1 > p_1 a_{11} + p_2 a_{21} + w l_1$$
$$p_2 > p_1 a_{12} + p_2 a_{22} + w l_2$$

imply

$$p_1 > p_1 a_{11} + p_2 a_{21}$$
$$p_2 > p_1 a_{12} + p_2 a_{22}$$

Conversely if we have

$$p_1 > p_1 a_{11} + p_2 a_{21}$$
$$p_2 > p_1 a_{12} + p_2 a_{22}$$

then society is viable if w is small and p_3 is large enough.
But

$$p_1 > p_1 a_{11} + p_2 a_{21}$$
$$p_2 > p_1 a_{12} + p_2 a_{22}$$

can be written as $p > pA$, i.e. ${}^tA$ is productive. We have the following result

Proposition 6 *The capitalist society is viable iff the technology ${}^tA$ is productive, or equivalently A is productive.*

The proof is immediate.

An example
Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ with $a_{ij} > 0, \forall i, j$. We assume $a_{11} + a_{12} < 1,\ a_{21} + a_{22} < 1$.

- Show that $(I-A)^{-1} = \sum_{n=0}^{+\infty} A^n$.

- Show that A is productive

- Give a proof that the values $(\lambda_i), i = 1, 2, 3$, are strictly positive if l_1, l_2, l_3 are strictly positive.

- Give an interpretation of the conditions $a_{11} + a_{12} < 1,\ a_{21} + a_{22} < 1$.

What happens when l_1 decreases?

Suppose that l_1 passes to $l_1^* < l_1$ while the labour-time l_2 remains unchanged.Marx could not give a definite answer. Here we can with the mathematical formalization of Marx's model.

In the following propositions, we assume that $A >> 0, 1 - a_{11} > 0, 1 - a_{22} > 0$, $(I - A)$ is invertible, the labour times (l_1, l_2, l_3) are strictly positive.

Proposition 7 *Suppose that l_1 falls by $dl_1 < 0$. Then λ_1, λ_2 fall too, i.e. λ_1 passes to $\lambda_1^* < \lambda_1$ and λ_2^* passes to $\lambda_2^* < \lambda_2$.*
Moreover

$$\frac{\lambda_2^*}{\lambda_2} > \frac{\lambda_1^*}{\lambda_1}$$

(the relative fall of λ_2 is smaller than the fall of λ_1.)

Proof: The falls of λ_1, λ_2 satisfy the equations

$$\begin{pmatrix} d\lambda_1 \\ d\lambda_2 \end{pmatrix} =^t A \begin{pmatrix} d\lambda_1 \\ d\lambda_2 \end{pmatrix} + \begin{pmatrix} dl_1 \\ 0 \end{pmatrix}$$

Tedious computations give

$$d\lambda_2 = \frac{a_{12} d\lambda_1}{(1 - a_{22})}$$
$$[(1 - a_{11})(1 - a_{22}) - a_{12}a_{21}]d\lambda_1 = (1 - a_{22})dl_1$$

From the proof of Lemma 2, $(1 - a_{11})(1 - a_{22}) - a_{12}a_{21} > 0$, $(1 - a_{22}) > 0$. Hence $d\lambda_1 < 0, d\lambda_2 < 0$.

Now

$$\frac{\lambda_2^*}{\lambda_2} = \frac{a_{12}\lambda_1^* + a_{22}\lambda_2^* + l_2}{a_{12}\lambda_1 + a_{22}\lambda_2 + l_2}$$

Set $\lambda_1^* = \rho\lambda_1, \lambda_2^* = \mu\lambda_2$, $\rho < 1, \mu < 1$. We then get

$$\mu[a_{12}\lambda_1 + a_{22}\lambda_2 + l_2] = a_{12}\rho\lambda_1 + a_{22}\mu\lambda_2 + l_2$$

and then

$$(\mu - \rho)a_{12}\lambda_1 = l_2(1 - \mu) > 0$$

Thus $\mu > \rho$. ■

Obviously, we have also this result

Proposition 8 *Suppose that l_2 falls by $dl_2 < 0$. Then λ_1, λ_2 fall too, i.e. l_1 passes to $\lambda_1^* < \lambda_1$ and λ_2^* passes to $\lambda_2^* < \lambda_2$.*
Moreover

$$\frac{\lambda_1^*}{\lambda_1} > \frac{\lambda_2^*}{\lambda_2}$$

(the relative fall of λ_1 is smaller than the fall of λ_2.)

Proposition 9 *If the coefficient a_{11} decreases (the production technology for capital good 1 is more efficient) then the values λ_1, λ_2 decreases. The relative diminution of λ_1 is lower than the relative diminution of λ_2.*
If the coefficient a_{22} decreases (the production technology for capital good 2 is more efficient) then the values λ_1, λ_2 decreases. The relative diminution of λ_2 is lower than the relative diminution of λ_1.

Proof: The values λ_1, λ_2 solve the system (4). Tedious computations give

$$\lambda_1[(1-a_{11})(1-a_{22}) - a_{12}a_{21}] = a_{21}l_2 + (1-a_{22})l_1$$
$$\lambda_2 = \frac{a_{12}}{(1-a_{22})}\lambda_1 + \frac{l_2}{1-a_{22}}$$

Recall that from our assumptions $D = [(1-a_{11})(1-a_{22})-a_{12}a_{21}] > 0, 1-a_{22} > 0$. Suppose $da_{11} < 0$. One gets

$$\frac{d\lambda_1}{\lambda_1} = \frac{1-a_{22}}{D}da_{11} \Rightarrow d\lambda_1 < 0 \Rightarrow d\lambda_2 < 0$$

Since

$$\frac{\lambda_2^*}{\lambda_2} = \frac{a_{12}\lambda_1^* + a_{22}\lambda_2^* + l_2}{a_{12}\lambda_1 + a_{22}\lambda_2 + l_2}$$

set $\lambda_1^* = \rho\lambda_1, \lambda_2^* = \mu\lambda_2,\ \rho < 1, \mu < 1$. We then get

$$\mu[a_{12}\lambda_1 + a_{22}\lambda_2 + l_2] = a_{12}\rho\lambda_1 + a_{22}\mu\lambda_2 + l_2$$

and then

$$(\mu - \rho)a_{12}\lambda_1 = l_2(1-\mu) > 0$$

Thus $\mu > \rho$.
Similarly, we get the second result. ∎

About the 'organic composition of capital'
We quote Marx (*Capital*, vol.1, p.612)): ' *The composition of capital is to be understood in two-fold sense. On the side of value, it is determined by the proportion in which it is divided into constant capital or value of the means of production, and variable capital or value of labour-power, the sum of total wages. On the side of material, as it functions in the process of production, all*

capital is divided into means of production and living labour-power. This latter composition is determined by the relation between the mass of production employed, on the one hand, and the mass of labour necessary for their employment on the other. I call the former the value-composition, the latter the technical composition of capital. Between the two there is a strict correlation. To express this, I call the value-composition of capital, in so far as it is determined by its technical composition and mirrors the changes of the latter, the organic composition of capital'.

Let b be the amount of wage good which is necessary to produce one unit of labour-power. Then the value-compositions are

$$\text{for good 1, } k_1 = \frac{a_{11}\lambda_1 + a_{21}\lambda_2}{b\lambda_3 l_1}$$
$$\text{for good 2, } k_2 = \frac{a_{12}\lambda_1 + a_{22}\lambda_2}{b\lambda_3 l_2}$$
$$\text{for good 3, } k_3 = \frac{a_{13}\lambda_1 + a_{23}\lambda_2}{b\lambda_3 l_3}$$

Recall we have

$$(\lambda_1 \ \ \lambda_2) = (\lambda_1 \ \ \lambda_2)^t A + (l_1 \ \ l_2) \Leftrightarrow (\lambda_1 \ \ \lambda_2) = (l_1 \ l_2)(I -^t A)^{-1}$$
$$\lambda_3 = (\lambda_1 \ \ \lambda_2)\begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix} + l_3 \Leftrightarrow \lambda_3 = (l_1 \ l_2)(I -^t A)^{-1}\begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix} + l_3$$

Define

$$\delta_i = b\lambda_3 l_i, \quad i = 1, 2, 3$$

We get

$$\delta_i = bl_i[(l_1 \ l_2)(I -^t A)^{-1}\begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix} + l_3]$$

Define also

$$n_1 = a_{11}\lambda_1 + a_{21}\lambda_2 = (l_1 \ l_2)(I -^t A)^{-1}\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$$
$$n_2 = a_{12}\lambda_1 + a_{22}\lambda_2 = (l_1 \ l_2)(I -^t A)^{-1}\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$$
$$n_3 = a_{13}\lambda_1 + a_{23}\lambda_2 = (l_1 \ l_2)(I -^t A)^{-1}\begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix}$$

We then obtain

$$\text{for good 1, } k_1 = \frac{n_1}{\delta_1}$$
$$\text{for good 2, } k_2 = \frac{n_2}{\delta_2}$$
$$\text{for good 3, } k_3 = \frac{n_3}{\delta_3}$$

That are the relations between the value-compositions and technical compositions of capital.

We quote Marx (*Capital*, vol.1, p.35,p.36, p.41, 85, p.87):
' A commodity is, in the first place, an object outside us, a thing that by its properties satisfies human wants of some sort or another. The nature of such wants, whether, for instance, they spring from the stomach or some fancy, makes no difference.' 'The utility of a thing makes it a use-value...This property of a commodity is independent of the amount of labour required to appropriate its useful qualities...'
'In order to produce the latter [commodities], he must not only produce use-values, but use-values for others, social use-values...Lastly nothing can have value, without being an object of utility.'
His commodity posses for himself [the owner of the commodity] no immediate use-value. Otherwise, he would not bring it to the maeket. It has use-value for others; but for himself its only direct use-value is that being a depository of exchange-value, and consequently, a means of exchange. All commodities are non use-values for their owners, and use-values for their non-owners. Consequently, they must all change hands.'
Because of these writings of Marx, Morishima believes that 'Marx would have accepted the marginal utility theory of consumer's demand if it had become known to him.'
Let us accept this belief of Morishima.

Equilibrium
We consider an individual with utility function $u(x_1, x_2, x_3)$. Her/his initial endowments of goods are $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$. Let ν_i denote the non-use-value of commodity i for this individual. We suppose that $\nu_i = \nu p_i$ where p_i is the exchange value of commodity i. The individual with maximize the sum of utililty and the total of non-use-value which is

$$u(x_1, x_2, x_3) + \nu p_1(\bar{x}_1 - x_1) + \nu p_2(\bar{x}_2 - x_2) + \nu p_3(\bar{x}_3 - x3)$$

We suppose the utility function is strictly concave, strictly increasing and satisfies Inada conditions. The FOC are

$$\frac{u_1}{p_1} = \frac{u_2}{p_2} = \frac{u_3}{p_3} = \nu \tag{8}$$

We make the following assumption: there exists no 'exploitation' of the workers or equivalently the labourers are fully paid, i.e.

$$p_1 = p_1 a_{11} + p_2 a_{21} + w l_1 \tag{9}$$
$$p_2 = p_1 a_{12} + p_2 a_{22} + w l_2 \tag{10}$$
$$p_3 = p_1 a_{13} + p_2 a_{23} + w l_3 \tag{11}$$

Comparing with (4) and since the matrix A is productive we obtain

$$\frac{p_1}{w} = \lambda_1, \quad \frac{p_2}{w} = \lambda_2, \quad \frac{p_3}{w} = \lambda_3 \tag{12}$$

Let N be the total number of labourers. The consumption at subsistence level of the labourers is exogenously given and equals b. Each individual works T hours a day. The level of consumption of the worker is $\beta = \frac{wT}{p_3 b}$. The total demand for wage good is

$$D = N\beta b = N\frac{wT}{p_3}$$

The equilibrium condition between demand and supply for wage is

$$x_3 = D \tag{13}$$

. The production of x_3 amount of wage good induces a production of capital goods:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + x_3 \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix} \tag{14}$$

Recall the equations determining the values

$$(\lambda_1 \;\; \lambda_2) = (l_1 \;\; l_2)(I -^t A)^{-1} \tag{15}$$

$$\lambda_3 = \lambda_1 a_{13} + \lambda_2 a_{23} + l_3 \tag{16}$$

The last equation is the total demand for labor equals the total supply

$$x_1 l_1 + x_2 l_2 + x_3 l_3 = NT \tag{17}$$

The equations (13), (14) determine $x_1(\lambda_3), x_2(\lambda_3), x_3(\lambda_3)$. The system composed by (9), (10), (11) associated with (12) is equivalent to the system composed by (15), (16). Finally, equation (17) gives the value of λ_3. Jointly with (15), (16) we determine λ_1, λ_2. If we fix $w = 1$ we determine p_1, p_2, p_3 by using (17) and (12).

The Theory of Exploitation

Chapter 3: Surplus value and exploitation

We cite here Morishima: " Marx considered exploitation as necessary for the maintenance of capitalist society. In fact, capitalists exploit workers by making them work longer than the hours required to produce the amounts of wage goods which they can buy with the wages they receive; thus surplus outputs are produced, which are source of profits. As capitalists would not be interested in their enterprises if they did not bring forth positive profits, they have a fundamental tendency towards exploitation".

What is exploitation? There are three definitions of rate exploitation. We will show that they are equal.
Rate of exploitation: definition 1
Let b denote the daily means of subsistence of the labourer. Let T be the prevailing length of the working day. The labour-time to obtain b is $\lambda_3 b$. The first definition of rate of exploitation is

$$e_1 = \frac{\textit{Unpaid labour}}{\textit{Paid labour}} = \frac{T - \lambda_3 b}{\lambda_3 b} = \frac{1 - \omega\lambda_3 b}{\omega\lambda_3 b}$$

where $\omega = \frac{1}{T}$.
We cite Marx: ' On the basis of capitalist production...this necessary labour $[\lambda_3 b]$ can form a part only of the working-day $[T]$ the working-day itself can never be reduced to this minimum (*Capital*, vol.1, p.232.) Thus Marx made the basic assumption of the theory of exploitation: $T > b\lambda_3$.

Rate of exploitation: definition 2
Let $\bar{N}$ labourers who work T hours a day. They constitute the labour-power. In order to maintain them at the subsistence level, they must produce $b\bar{N}$ amounts of wage goods per day. The capital goods required are $b\bar{N}\begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix}$. These demands for capital goods from wage-good sector have repercussions on the capital good industries. The capital goods $\bar{x}_1, \bar{x}_2$ will be produced:

$$\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} = (I - A)^{-1} b\bar{N} \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix}$$

Let N be the number of necessary labourers. The total labourer-time required for the production of the wage goods is

$$TN = l_1\bar{x}_1 + l_2\bar{x}_2 + l_3 b\bar{N}$$

The surplus of labourers $\bar{N} - N$ are unnecessary and work for the benefit of the capitalists. The second definition of the rate of exploitation is

$$e_2 = \frac{T\bar{N} - TN}{TN} = \frac{\textit{Total surplus labour}}{\textit{Socially necessary labour}}$$

Let us compute the value of e_2. We have

$$\begin{aligned} TN = &\quad (l_1 \;\; l_2) \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} + b\bar{N}l_3 \\ = &\quad (l_1 \;\; l_2)[(I-A)^{-1}b\bar{N} \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix}] + b\bar{N}l_3 \\ = &\quad (l_1 \;\; l_2)b\bar{N}[I-A)^{-1} \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix} + l_3] \end{aligned}$$

Since

$$\begin{aligned} \lambda_3 = &\quad (\lambda_1 \;\; \lambda_2) \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix} + l_3 \\ = &\quad (l_1 \;\; l_2)(I-A)^{-1} \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix} + l_3 \end{aligned}$$

We get

$$TN = b\lambda_3\bar{N}$$

and

$$e_2 = \frac{T\bar{N} - TN}{TN} = \frac{1 - \omega\lambda_3 b}{\omega\lambda_3 b} = e_1$$

Rate of exploitation: definition 3
The third definition of rate of exploitation is

$$e_3 = \frac{\textit{Total surplus value}}{\textit{Total value of labour-power}}$$

We will explicit the total surplus value and the total value of labour-power. The labour power is represented by the number $\bar{N}$ of labourers. For the subsistence level of them, the wage-good industries must produce $b\bar{N}$ wage-goods and outputs (x_1, x_2) of capital goods, x_3 of wage good. The total employment is

$$T\bar{N} = x_1 l_1 + x_2 l_2 + x_3 l_3$$

Th capital goods required are

$$x_1^* = a_{11}x_1 + a_{12}x_2 + a_{13}x_3$$
$$x_2^* = a_{21}x_1 + a_{22}x_2 + a_{23}x_3$$

The surplus products of capital goods are given by $(x_1 - x_1^*, x_2 - x_2^*)$ and for wage good $(x_3 - b\bar{N})$.
The surplus value is

$$\lambda_1(x_1 - x_1^*) + \lambda_2(x_2 - x_2^*) + \lambda_3(x_3 - b\bar{N})$$

The total value of labour-power is $\lambda_3 b\bar{N}$.
Recall that

$$\lambda_3 = a_{13}\lambda_1 + a_{23}\lambda_2 + l_3$$
$$\lambda_1 = a_{11}\lambda_1 + a_{21}\lambda_2 + l_1$$
$$\lambda_2 = a_{12}\lambda_1 + a_{22}\lambda_2 + l_2$$

Tedious computations yield

$$\lambda_1(x_1 - x_1^*) + \lambda_2(x_2 - x_2^*) + \lambda_3(x_3 - b\bar{N}) = x_1 l_1 + x_2 l_2 + x_3 l_3 - \lambda_3 b\bar{N} = T\bar{N} - \lambda_3 b\bar{N}$$

Hence

$$e_3 = \frac{T\bar{N} - \lambda_3 b\bar{N}}{\lambda_3 b\bar{N}} = \frac{1 - \omega\lambda_3 b}{\omega\lambda_3 b} = e$$

Recall $\omega = \frac{1}{T}$.
The following assumption is very important: The wage rate w is at least as high as the subsistence level, so that the labourer can buy ωb amounts of wage good by spending his hourly wages, i.e.

$$Tw \geq p_3 b$$

Proposition 10 *Assume $Tw \geq p_3 b$.*
The capitalist system is viable iff there is exploitation.

Proof: (i) Suppose the capitalist system is viable, i.e. there exists prices and wage system $(p_1, p_2, p3, w)$ for which every industry makes profits

$$(p_1 \;\; p_2) > (p_1 \;\; p_2)A + w(l_1 \;\; l_2)$$
$$p_3 > p_1 a_{13} + p_2 a_{23} + w l_3$$

Since $w \geq p_3 \omega b$ we have

$$\begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix} > \begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{13} \\ \omega b l_1 & \omega b l_2 & \omega b l_3 \end{pmatrix}$$

Therefore there exist $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ s.t.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} > \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{13} \\ \omega b l_1 & \omega b l_2 & \omega b l_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

and

$$\begin{pmatrix}\lambda_1 & \lambda_2 & \lambda_3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} > \begin{pmatrix}\lambda_1 & \lambda_2 & \lambda_3\end{pmatrix}\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{13}\\ \omega b l_1 & \omega b l_2 & \omega b l_3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

From this inequality one gets

$$0 < (\begin{pmatrix}\lambda_1 & \lambda_2\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} + \lambda_3 x_3) - \begin{pmatrix}\lambda_1 & \lambda_2\end{pmatrix}(A\begin{pmatrix}x_1\\x_2\end{pmatrix} + x_3\begin{pmatrix}a_{13}\\a_{23}\end{pmatrix}) - \lambda_3\omega b(l_1x_1 + l_2x_2 + l_3x_3)$$
$$= e\omega\lambda_3 b(l_1x_1 + l_2x_2 + l_3x_3)$$

Hence $e > 0$.
(ii) To prove the converse. Assume $e > 0$.
Observe, since we have

$$\begin{pmatrix}\lambda_1 & \lambda_2\end{pmatrix} = \begin{pmatrix}\lambda_1 & \lambda_2\end{pmatrix}A + (l_1 \;\; l_2)$$
$$\lambda_3 = \lambda_1 a_{13} + \lambda_2 a_{23} + l_3$$
$$(1+e)\omega b\lambda_3 = 1$$

We get

$$\begin{pmatrix}\lambda_1 & \lambda_2\end{pmatrix} > \begin{pmatrix}\lambda_1 & \lambda_2\end{pmatrix}A + \lambda_3\omega b(l_1 \;\; l_2)$$
$$\lambda_3 > \lambda_1 a_{13} + \lambda_2 a_{23} + \lambda_3\omega b l_3$$

Take $(p_1 \;\; p_2 \;\; p_3) = (\lambda_1 \;\; \lambda_2 \;\; \lambda_3)$ and $w = \lambda_3\omega b$. We see that the capitalist system is viable with this system (p_1, p_2, p_3, w)(the profits are positive in any sector). ■

To sum up: to have a positive rate of exploitation, the capitalist society must be viable. This implies a productive technology. The value of wage good must be low enough to make the value of means of subsistence $\lambda_3 b$ less than the working-day T.

We consider a long-run equilibrium. The rates of profit are equalized in all the sectors. Denote by π this rate of profit. We have

$$\begin{aligned} &\begin{pmatrix} p_1 & p_2 \end{pmatrix} = (1+\pi)(\begin{pmatrix} p_1 & p_2 \end{pmatrix} A + w \begin{pmatrix} l_1 & l_2 \end{pmatrix}) \\ &p_3 = (1+\pi)(p_1 a_{13} + p_2 a_{23} + w l_3) \end{aligned}$$

Recall we have $w = p_3 \omega b$, where $\omega = \frac{1}{T}$.
Recall that $1 = (1+e)\lambda_3 \omega b$. The value-determining equations are

$$\begin{aligned} &\begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} A + [(1+e)\lambda_3 \omega b] \begin{pmatrix} l_1 & l_2 \end{pmatrix} \\ &\lambda_3 = a_{13}\lambda_1 + a_{23}\lambda_2 + [(1+e)\lambda_3 \omega b] l_3 \end{aligned}$$

We check that we have

$$\begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} < (1+e)[\begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} A + \lambda_3 \omega b \begin{pmatrix} l_1 & l_2 \end{pmatrix}] \tag{18}$$

$$\lambda_3 < (1+e)[\lambda_1 a_{13} + \lambda_2 a_{23} + \lambda_3 \omega b l_3] \tag{19}$$

Denote

$$\Lambda = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix}$$

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \omega b l_1 & \omega b l_2 & \omega l_3 \end{pmatrix}$$

The system (18), (19) can be wriitten as

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} < (1+e) \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} M$$

Lemma 4 *There exists no $x \in \mathbb{R}^3_+ \setminus \{0\}$ which satisfies $x = (1+f)Mx$ for some $f \geq e$.*

Proof: Indeed if such an x exists, then we have a contradiction

$$\Lambda x < (1+e)\Lambda M x \leq (1+f)\Lambda M x = \Lambda x$$

■ Observe we have the following price-determining equations

$$\begin{aligned} &\begin{pmatrix} p_1 & p_2 \end{pmatrix} = (1+\pi)[\begin{pmatrix} p_1 & p_2 \end{pmatrix} A + p_3 \omega b \begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix}] \\ &p_3 = (1+\pi)[p_1 a_{13} + p_2 a_{23} + p_3 \omega b l_3] \end{aligned}$$

or in more compact form

$$\begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix} = (1+\pi) \begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix} M \tag{20}$$

We now establish Marx's main result.

Proposition 11 *Then the rate of profit is less than the rate of exploitation e.*

Proof: Lets $x \in \mathbb{R}^3$ satisfy $x = (1+\pi)Mx$.
Assume the contrary $\pi \geq e$. From Lemma 4, this vector x must contain negative components. Let x^* denote the vector by replacing the negative components of x by 0. Then, $Mx^* \geq 0, Mx^* \geq Mx$. Then

$$x^* \leq (1+\pi)Mx^*$$

Actually, by Lemma (4) we have

$$x^* < (1+\pi)Mx^*$$

Hence

$$\begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix} x^* < (1+\pi) \begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix} Mx^*$$

However from (20) we have

$$\begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix} x^* = (1+\pi) \begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix} Mx^*$$

a contradiction. Hence $\pi < e$.
In the state of equilibrium, workers are inevitably exploited by capitalists with a rate larger than the rate of profit. ∎

Proposition 12 *Suppose that no sector makes negative profit. In this case we have*

$$p_1/w \geq \lambda_1, p_2/w \geq \lambda_2$$

Proof: We have

$$(p_1/w \quad p_2/w)(I-A) \geq (l_1 \ \ l_2)$$

Since the matrix A is productive, $(I-A)^{-1}$ is strictly positive. We then have

$$(p_1/w \ \ p_2/w) \geq (l_1 \ \ l_2)(I-A)^{-1}$$

Recall we also have

$$(\lambda_1 \ \ \lambda_2) = (l_1 \ \ l_2)(I-A)^{-1}$$

Hence $p_1/w \geq \lambda_1, p_2/w \geq \lambda_2$. ∎

The following result is not mentioned in Morishima. Is it in Marx's Capital III?

Proposition 13 *Assume that no sector makes negative profit. Then* $e = \pi \Leftrightarrow \pi = e = 0$

Proof: Recall

- $Tw \geq p_3 b \Leftrightarrow 1 \geq \frac{p_3 b \omega}{w}$
- $\lambda_3 = \lambda_1 a_{13} + \lambda_2 a_{23} + l_3$
- $1 + \pi = \frac{p_3}{p_1 a_{13} + p_2 a_{23} + w l_3}$

Therefore

$$\begin{aligned} 1+\pi &= \frac{p_3}{p_1 a_{13} + p_2 a_{23} + w l_3} \\ &\leq \frac{(p_3/w)}{\lambda_1 a_{13} + \lambda_2 a_{23} + l_3} \leq \frac{p_3}{w\lambda_3} \leq \frac{1}{\omega \lambda_3 b} = 1 + e \end{aligned}$$

If $e = \pi$ then

$$\begin{aligned} 1+\pi &= \frac{p_3}{p_1 a_{13} + p_2 a_{23} + w l_3} \\ &= \frac{(p_3/w)}{\lambda_1 a_{13} + \lambda_2 a_{23} + l_3} = \frac{p_3}{w\lambda_3} = \frac{1}{\omega \lambda_3 b} = 1 + e \end{aligned}$$

We claim that $p_1/w = \lambda_1, p_2/w = \lambda_2$. Indeed, suppose $p_1/w > \lambda_1$. In this case we have a contradiction

$$\begin{aligned} 1+\pi &= \frac{p_3}{p_1 a_{13} + p_2 a_{23} + w l_3} \\ &< \frac{(p_3/w)}{\lambda_1 a_{13} + \lambda_2 a_{23} + l_3} = \frac{p_3}{w\lambda_3} = \frac{1}{\omega \lambda_3 b} = 1 + e \end{aligned}$$

This implies $\pi < e$.
Since

$$\begin{pmatrix} p_1/w & p_2/w \end{pmatrix} = (1+\pi)(\begin{pmatrix} p_1/w & p_2/w \end{pmatrix} A + \begin{pmatrix} l_1 & l_2 \end{pmatrix})$$

and $p_1/w = \lambda_1, p_2/w = \lambda_2$ we obtain

$$\begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} = (1+\pi)(\begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} A + \begin{pmatrix} l_1 & l_2 \end{pmatrix})$$

But the values determination equation gives

$$\begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} A + (l_1 \ \ l_2)$$

Gather the two previous equations:

$$\begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} = (1+\pi)(\begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix})$$

This implies $\pi = 0$. Since we assume $e = \pi$, hence $e = 0$. ■

Chapter 5: The Law of Falling Rate of profit

"The gradual growth of constant capital in relation to variable capital must necessaily lead to a *gradual fall of the general rate of profit*, so long as the rate of surplus-value, or the intensity of exploitation of labour by capital, remain the same' (*Capital*, III, p.212)
Let e denote the rate of exploitation. Recall: $(1+e)\omega\lambda_3 b = 1$, where $\omega = \frac{1}{T}$. Define the constant capitals $C_1, C_2, C_3,\ C_1^p, C_2^p, C_3^p$ and the variable capitals $V_1, V_2, V_3, V_1^p, V_2^p, V_3^p$.

$$\text{For } i = 1,2,3, \qquad C_i = \lambda_1 a_{1i} + \lambda_2 a_{2i}, C_i^p = p_1 a_{1i} + p_2 a_{2i}$$
$$V_i = \omega\lambda_3 b l_i, V_i^p = \omega p_3 b l_i$$

We define the profits

$$\Pi_i = p_i - (C_i^p + V_i^p)$$

and the surplus

$$S_i = \lambda_i - (C_i + V_i) = eV_i$$

To understand the last equalities, recall

$$\lambda_i = C_i + l_i = C_i + (\lambda_3\omega b)l_i + (e\lambda_3\omega b)l_i = C_i + V_i + S_i$$

with $S_i = (e\lambda_3\omega b)l_i$.

Lemma 5 *Assume $wT = p_3 b$, i.e. wages are fixed a a level at which workers can only purchase the daily means of subsistence b. We assume also that workers must work T hours a day.*
Assume $\frac{S_1}{\Pi_1} = \frac{S_2}{\Pi_2} = \frac{S_3}{\Pi_3} = \alpha$.
Then $\lambda_i = \alpha p_i$, for every $i = 1,2,3$.

For a proof, refer to Marx's Economics by Michio Morishima, p.73

Proposition 14 (The law of falling rate of profit) *Assume*

- $wT = p_3 b$
- $\frac{S_1}{\Pi_1} = \frac{S_2}{\Pi_2} = \frac{S_3}{\Pi_3} = \alpha$
- *The exploitation rate e is constant.*

Then the rate of profit in sector i decreases if $\frac{C_i}{V_i}$ increases.

Proof: Under these assumptions we have $C_i = \alpha C_i^p, V_i = \alpha V_i^p$. Therefore

$$\pi_i = \frac{\Pi_i}{C_i^p + V_i^p + S_i^p} = \frac{S_i}{C_i + V_i + S_i}$$
$$= \frac{eV_i}{C_i + (1+e)V_i} = \frac{e}{C_i/V_i + (1+e)}$$

Hence π_i decreases if $\frac{C_i}{V_i}$ increases. ■

Appendix

Let $A = (a_{ij}), i = 1, \ldots, n; j = 1, \ldots, n$ be a $n \times n$ matrix which satisfies $a_{ij} > 0, \forall i, j$.
We say that A is productive if there exists $p \in \mathbb{R}^n_{++}$ such that $p > Ap$ in the sense the vector $p - Ap$ has strictly positive components.
The proof of the following lemma is borrowed from Wikipedia.

Lemma 6 *A is productive iff the matrix $(I - A)^{-1}$ exists and has no negative coefficients.*

Proof: (i) We prove: if $(I - A)^{-1}$ exists and has no negative coefficients, then A is productive.
Indeed, take $q \in \mathbb{R}^n_{++}$. Let $p = (I - A)^{-1}q$. Then $p >> 0$. We have $(I - A)p = q > 0$. Thus A is productive.
(ii) Now suppose A productive and $(I - A)^{-1}$ does not exist. Let $p \in \mathbb{R}^n_{++}$ satisfy $p > Ap$. Define $v = p - Ap$. Then $v_k > 0, \forall k$. Let $Z = \ker(I - A)$. There must exist $z \in Z$ with $z_i > 0$.
Define $c = \sup_i \frac{z_i}{p_i}$. We can find k such that $c = \frac{z_k}{p_k}$. Then

$$0 < cv_k = cp_k - \sum_{i=1}^{n} a_{ki} c p_i = z_k - \sum_{i=1}^{n} a_{ki} c p_i \leq z_k - \sum_{i=1}^{n} a_{ki} z_i = 0$$

We have a contradiction. Hence $I - A$ is invertible.
We now prove that the coefficients of $(I - A)^{-1}$ are non negative. Suppose the contrary. This matrix has a negative coefficient. In his case there exists a vector $x \geq 0$ such that the vector $y = (I - A)^{-1}x$ has at least a negative component. Define

$$c = \sup_i \frac{-y_i}{p_i}$$

Suppose $c = \frac{-y_k}{p_k}$. We have

$$0 < cv_k = c(p_k - \sum_{i=1}^{n} a_{ki} p_i) \leq -y_k + \sum_{i=1}^{n} a_{ki} y_i = -x_k \leq 0$$

We have a contradiction. Hence, the coefficients of $(I - A)^{-1}$ are non negative.
■