Walras and Dividends Equilibrium With
Possibly Satiated Consumers

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Abstract

The main contribution of the paper is to provide a weaker non-satiation assumption than the one commonly used in the literature to ensure the existence of competitive equilibrium. Our assumption allows for satiation points in the set of individually feasible consumptions, provided that the consumer has satiation points available to him outside this set. As a result, we show the concept of equilibrium with dividends (See Aumann and Dreze (1986), Mas-Collel (1992)) is pertinent only when the set of satiation points is included in the set of individually feasible consumptions. Our economic motivation stems from the fact that in decentralized markets, increasing the incomes of consumers through dividends, if it is possible, is costly since it involves the intervention of a social planner. Then, we show, in particular, how in securities markets our weak nonsatiation assumption is satisfied by Werner's (1987) assumption.

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1 Introduction

Since the seminal contributions of Arrow-Debreu (1954), and Mckenzie (1959), on the existence of a competitive equilibrium, a subject of ongoing interest in the economics profession has been the robustness of the various assumptions made to ensure such a result. On the consumer side, assumptions such as, the convexity of preferences, free-disposal and survival have been investigated both conceptually and empirically by numerous economists ranging from development economists to decision theorists. The seemingly innocuous assumption of nonsatiation, normally represented in Microeconomics textbooks by the monotonicity of preferences, appears to have received much less attention. Perhaps, the main critique to the insatiability assumption is that the human nature calls it into question. Namely, any moderately greedy person will testify to their occasional satiation. Technically, a satiation point seems to be genuinely guaranteed with continuous preferences, whenever the choice set is bounded. Having a bounded choice set is hardly surprising, as consumption activities take place over a limited time span. Accordingly, this condition has been weakened by assuming that nonsatiation holds only over individually feasible consumptions;[1] that is to say, satiation levels are higher than the actual consumption levels involved in trade. In the presence of satiation points in individually feasible consumption sets, we find in the literature the concept of equilibrium with coupons or dividends that extend the classical general equilibrium theory to the class of such economies (see Aumann and Dreze (1986), Mas-Collel (1992), Kaji (1996), Cornet, Topuzu and Yildiz (2003)).[2] The underlying idea is to allow the nonsatiated consumers to benefit, through dividends, from the budget surplus created by non budget-binding optimal consumptions of satiated consumers. The analysis of the above-named authors has proved to be relevant to the study of markets with price rigidities, such as Labor market. One issue with equilibrium with dividends is that, increasing the incomes of consumers in decentralized markets, if it is possible, is costly since it involves the intervention of a social planner. In this paper, our main contribution is to introduce a weak nonsatiation assumption that ensures the existence of an exact competitive equilibrium

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[1]Individually feasible consumptions are defined as the projection, on individual's choice set, of consumptions that could potentially be achieved by trade

[2]See also Dreze and Muller (1980), Florig and Yildiz (2002), Le Van and Minh (2004), Konovalov (2005).

(without dividends). Our assumption allows for the satiation points in individually feasible consumption set, provided that the satiation area is not a subset of the individually feasible consumptions. Stated formally, if we consider M_i to be the maximum of the utility function of consumer i over individually feasible consumption set, our assumption stipulates that there be a consumption bundle outside the individually feasible consumption set that guarantees at least the utility level M_i. The standard nonsatiation assumption rather requires that there be a consumption bundle outside the individually feasible consumption set that guarantees strictly more than the utility level M_i. In a recent paper, Yannelis and Won (2005) demonstrate the existence of a competitive equilibrium with a different nonsatiation assumption, in a more general setting. Their assumption allows the satiation area to be inside the individually feasible consumption sets, provided that it is unaffordable with respect to any price system supporting the preferences of the nonsatiated consumers. Yannelis and Won (2005) also show that their assumption is implied by our weak nonsatiation assumption and could be suitably applied to some asset pricing models. Notwithstanding the novelty of their approach, their assumption relies on price systems, whereas our weak nonsatiation is defined on the primitives of the economy. In securities markets with short-selling, Werner (1987) introduces a nonsatiation assumption which allows the existence of satiation points even if they are in the projections of the feasible set. Werner's assumption stipulates that each trader has a useful portfolio. This is defined as a portfolio which, when added, at any rate, to any given portfolio increases the trader's utility. In particular his assumption implies an unbounded set of satiation point. In the paper, we show that Werner's nonsatiation implies our weak nonsatiation assumption. We also provide an example where Werner's nonsatiation does not hold, whereas our weak nonsatiation assumption is satisfied, and consequently an equilibrium exists. The paper is organized as follows, Section 2 is devoted to the model. In Section 3, we shall introduce our new nonsatiation assumption. In section 4, we compare our new nonsatition assumption with Werner's nonsatiation. Section 5 is an Appendix.

2   The Model

We consider an economy having a finite number l of goods and a finite number I of consumers. For each $i \in I$, let $X_i \subset R^l$ denote the set of consumption

goods, let $u_i : X_i \longrightarrow R$ denote the utility function and let $e_i \in X_i$ be the initial endowment. In the sequel, we will denote this economy by

$$\mathcal{E} = \{(X_i, u_i, e_i)_{i \in I}\}.$$

An *individually rational feasible allocation* is the list $(x_i)_{i \in I} \in \prod_{i \in I} X_i$, which satisfies $\sum_{i \in I} x_i = \sum_{i \in I} e_i$, and $u_i(x_i) \geq u_i(e_i), \ \forall i \in I$. We denote by $\mathcal{A}$ the set of individually rational feasible allocations. We shall denote by $\mathcal{A}_i$ the projection of $\mathcal{A}$ onto X_i.

The *set of individually rational utilities* is given by

$$\mathcal{U} = \{(v_i) \in R_+^{|I|} \mid \text{ there exists } x \in \mathcal{A} \text{ s.t. } u_i(e_i) \leq v_i \leq u_i(x_i), \ \forall i \in I\}.$$

In the following, for short, $\mathcal{U}$ will be called utility set.

We consider the following definition of Walras equilibrium (resp. quasi-equilibrium).

Definition 1 *A Walras equilibrium (resp. quasi-equilibrium) of $\mathcal{E}$ is a list $((x_i^*)_{i \in I}, p^*) \in \prod_{i \in I} X_i \times (R^l \setminus \{0\})$ which satisfies: (a) $\sum_{i \in I} x_i^* = \sum_{i \in I} e_i$ (Market clearing); (b) for each i one has $p^* \cdot x_i^* = p^* \cdot e_i$ (Budget constraint), and for each $x_i \in X_i$, with $u_i(x_i) > u_i(x_i^*)$, it holds $p^* \cdot x_i > p^* \cdot e_i$. [resp. $p^* \cdot x_i \geq p^* \cdot e_i$].*

In the presence of satiation points in individually feasible consumption sets, we find in the literature the concept of equilibrium with dividends (see Aumann and Dreze (1986), Mas-Collel (1992), Cornet, Topuzu and Yildiz (2003)). The dividends increase the income of nonsatiated consumers, in order to capture the surplus created by satiated consumers. In the following we define the concept of equilibrium (resp. quasi-equilibrium) with dividends.

Definition 2 *An equilibrium (resp. quasi-equilibrium) with dividends $(d_i^*)_{i \in I} \in R_+^{|I|}$ of $\mathcal{E}$ is a list $((x_i^*)_{i \in I}, p^*) \in \prod_{i \in I} X_i \times R^l$ which satisfies: (a) $\sum_{i \in I} x_i^* = \sum_{i \in I} e_i$ (Market clearing); (b) for each $i \in I$ one has $p^* \cdot x_i^* \leq p^* \cdot e_i + d_i^*$ (Budget constraint), and for each $x_i \in X_i$, with $u_i(x_i) > u_i(x_i^*)$, it holds $p^* \cdot x_i > p^* \cdot e_i + d_i^*$. [resp. $p^* \cdot x_i \geq p^* \cdot e_i + d_i^*$].*

When $d_i^* = 0$, for all $i \in I$, an equilibrium (resp. quasi-equilibrium) with dividends is a Walras equilibrium (resp. quasi-equilibrium).

Remark The passage from a quasi-equilibrium with dividends to an equilibrium with dividends is similar to the one in the standard Walrasian case. That is to say, let $((x_i^*)_{i \in I}, p^*)$ be a quasi-equilibrium with dividends $(d_i^*)_{i \in I}$. Assume that for all $i \in I$, the set $\{x_i \in X_i \mid u_i(x_i) > u_i(x_i^*)\}$ is relatively open in X_i, and $\inf p^* \cdot X_i < p^* \cdot x_i^*$, then, $((x_i^*)_{i \in I}, p^*)$ is an equilibrium with dividends $(d_i^*)_{i \in I}$.

Now, we list our assumptions:

(H_1) For each $i \in I$, the set X_i is closed and convex.

(H_2) For each $i \in I$, the utility function u_i is strictly quasi-concave and upper semicontinuous.[3]

(H_3) The utility set $\mathcal{U}$ is compact.

(H_4) For each $i \in I$, for all $x_i \in \mathcal{A}_i$, there exists $x_i' \in X_i$ such that $u_i(x_i') > u_i(x_i)$.

For every $i \in I$, let $S_i = \{x_i^* \in X_i : u_i(x_i^*) = \max_{x_i \in X_i} u_i(x_i)\}$. The set S_i is the set of satiation points for agent i. Observe that under assumptions $(H1) - (H2)$, the set S_i is closed and convex for every $i \in I$.

3 The Results

3.1 Equilibrium with dividends

We first give an existence of Walras quasi-equilibrium theorem when there exists no satiation.

Theorem 1 *Assume $(H_1) - (H_4)$, then there exists a Walras quasi-equilibrium.*

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[3]We recall that a function u_i is said to be quasi-concave if its level-set $L^\alpha = \{x_i \in X_i : u_i(x_i) \geq \alpha\}$ is convex, for each $\alpha \in R$.

The function u_i is strictly quasi-concave if and only if, for all $x_i, x_i' \in X_i$, $u_i(x_i') > u_i(x_i)$ and $\lambda \in [0, 1)$, then $u_i(\lambda x_i + (1 - \lambda)x_i') > u_i(x_i)$. It means that $u_i(\lambda x_i + (1 - \lambda)x_i') > \min(u_i(x_i), u_i(x_i'))$, if $u_i(x_i) \neq u_i(x_i')$. The function u_i is upper semicontinuous if and only if L^α is closed for each α.

Proof. The proof is quite standard. See e.g. Arrow and Debreu (1954) when the consumption sets are bounded from below, or the proof given in Dana, Le Van and Magnien (1999) for an exchange economy. □

Now, we come to our first result. This result has been proved by adding a virtual commodity to the economy and then modifying the utility functions of the agents. Our proof follows the steps of Le Van and Minh (2004) in introducing a new commodity, but our modification of the utility functions differs from theirs. The advantage of such modification will become clear when we introduce a new nonsatiation assumption. Let us recall the following definition.

Definition 3 *Let B be a closed convex nonempty set of R^l, where l is an integer. The recession cone of B, denoted by O^+B, is defined as follows:*

$$O^+B = \{w \in R^l : \forall x \in B, \forall \lambda \geq 0, x + \lambda w \in B.\}$$

We first give an intermediate result. The proof of the result is new, since we use a new modification of utility functions. The modified economy is, then, used to establish the existence of a quasi-equilibrium with dividends for the initial economy. In the following, we restrict the economy to compact consumption sets.

Proposition 1 *Assume $(H_1) - (H_2)$, and X_i is compact for every i. Then there exists a quasi-equilibrium with dividends.*

Proof. Let us introduce the auxiliary economy $\widehat{\mathcal{E}} = \left\{ (\widehat{X}_i, \widehat{u}_i, \widehat{e}_i)_{i \in I} \right\}$, where $\widehat{X}_i = X_i \times R_+$, $\hat{e}_i = (e_i, \delta_i)$ with $\delta_i > 0$ for any $i \in I$ and the utility functions $\widehat{u}_i$ are defined as follows. Let $M_i = \max \{u_i(x) \mid x \in \mathcal{A}_i\}$.

Case 1. If there exists $x_i^* \in \mathcal{A}_i^c$ (the complement of $\mathcal{A}_i$ in X_i), such that $u_i(x_i^*) > M_i$, then $\widehat{u}_i(x_i, d_i) = u_i(x_i)$ for every $(x_i, d_i) \in \widehat{X}_i$.

Case 2. Now, consider the case where there exists no $x_i \in \mathcal{A}_i^c$ which satisfies $u_i(x_i) > M_i$, but there exists $x_i^* \in \mathcal{A}_i^c$, such that $u_i(x_i^*) = M_i$. We modify agent's i utility function as follows: Using x_i^* we define the function

$$\lambda_i(\cdot) : S_i \to R_+ \cup \{+\infty\}$$

where,

$$\lambda_i(x_i) = \sup\{\beta \in R_+ \mid x_i^* + \beta(x_i - x_i^*) \in S_i\}.$$

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Now, using the function λ_i, we can define a new utility function, $\widehat{u}_i$, for agent i:

$$\widehat{u}_i(x_i, d_i) = \begin{cases} u_i(x_i) + 1 - \frac{1}{\lambda_i(x_i)}, & \text{if } x_i \in S_i \\ u_i(x_i), & \text{otherwise,} \end{cases}$$

for every $(x_i, d_i) \in \widehat{X}_i$.

Case 3. If there exists no $x_i \in \mathcal{A}_i^c$, such that $u_i(x_i) \geq M_i$, then, for some strictly positive μ_i let

$$\widehat{u}_i(x_i, d_i) = \begin{cases} u_i(x_i) + \mu_i d_i, & \text{if } x_i \in S_i \\ u_i(x_i), & \text{otherwise,} \end{cases}$$

for every $(x_i, d_i) \in \widehat{X}_i$. We will check that Assumption (H_2) is satisfied for every $\widehat{u}_i$. We will make use of the following lemma, the proof of which is in the Appendix.

Lemma 1 *Let B be a closed, convex set of R^n and x^* be in B. Let $\lambda : B \to R_+ \cup \{+\infty\}$ be defined by*

$$\lambda(x) = \sup\{\beta \geq 0 : x^* + \beta(x - x^*) \in B\}.$$

Then λ is upper semicontinuous and strictly quasi-concave.

To prove that $\widehat{u}_i$ is quasi-concave and upper semicontinuous, it suffices to prove that the set $\widehat{L}_i^\alpha = \{(x_i, d_i) \in X_i \times R_+ : \widehat{u}_i(x_i, d_i) \geq \alpha\}$ is closed and convex for every α.

Case 1. It is clear that $\widehat{L}_i^\alpha = L_i^\alpha \times R_+$, with $L_i^\alpha = \{x_i \in X_i : u_i(x_i) \geq \alpha\}$. Therefore, $\widehat{L}_i^\alpha$ is closed and convex for every α.

Case 2. (a) If $\alpha \leq M_i$, then obviously $\widehat{L}_i^\alpha = L_i^\alpha \times R_+$.
(b) If $M_i < \alpha \leq 1 + M_i$, then one can easily prove that $\widehat{L}_i^\alpha = \sigma^i(\frac{1}{(M_i+1-\alpha)}) \times R_+$, where $\sigma^i(\frac{1}{(M_i+1-\alpha)}) = \{x \in S_i : \lambda_i(x) \geq \frac{1}{(M_i+1-\alpha)}\}$. From Lemma 1 this set is closed and convex.
(c) If $1 + M_i < \alpha$, then obviously $\widehat{L}_i^\alpha = \emptyset$.

Case 3. We follow here the proof given by Le Van and Minh (2004), in which we have two cases,
(a) If $\alpha < M_i$. We claim that $\widehat{L}_i^\alpha = L_i^\alpha \times R_+$. Indeed, let $(x_i, d_i) \in \widehat{L}_i^\alpha$. It follows $\widehat{u}_i(x_i, d_i) \geq \alpha$ and there are two possibilities for x_i:

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- If $x_i \notin S_i$, then $\hat{u}_i(x_i, d_i) = u_i(x_i)$. It implies $u_i(x_i) \geq \alpha$ or $x_i \in L_i^\alpha$ and hence $(x_i, d_i) \in L_i^\alpha \times R_+$.

- If $x_i \in S_i$, then $u_i(x_i) = M_i > \alpha$. Thus, it follows that $x_i \in L_i^\alpha$ and $(x_i, d_i) \in L_i^\alpha \times R_+$.

So, $\hat{L}_i^\alpha \subset L_i^\alpha \times R_+$. But it is obvious that $L_i^\alpha \times R_+ \subset \hat{L}_i^\alpha$.

(b) If $\alpha \geq M_i$. We claim that $\hat{L}_i^\alpha = S_i \times \left\{ d_i \mid d_i \geq \frac{\alpha - M_i}{\mu} \right\}$. Indeed, if $\hat{u}_i(x_i, d_i) \geq \alpha$, then $x_i \in S_i$. In this case, $\hat{u}_i(x_i, d_i) = M_i + \mu d_i \geq \alpha$, and hence $d_i \geq \frac{\alpha - M_i}{\mu}$. The converse is obvious.

We have proved that $\hat{u}_i$ is upper semicontinuous and quasi-concave for every $i \in I$.

Now, we prove that $\hat{u}_i$ is strictly quasi-concave.

Claim 1 *The utility function $\hat{u}_i$ is strictly quasi-concave, for every $i \in I$.*

Proof. See the Appendix. □

Since X_i is assumed to be compact for every i, the individually feasible set $\hat{\mathcal{A}}$ of economy $\hat{\mathcal{E}}$ is also compact and Assumption (H_3) is fulfilled by economy $\hat{\mathcal{E}}$.

We now prove that the $\hat{u}_i$ has no satiation point on the set $\hat{\mathcal{A}}_i$, the projection of $\hat{\mathcal{A}}$ onto $\widehat{X}_i$.

Case 1. It is obvious.

Case 2. Since $\lambda_i(x_i^*) = +\infty$, it suffices to prove that $\lambda_i(x_i) < +\infty$, for any $x_i \in \mathcal{A}_i$. For that, take $x_i \in S_i \cap \mathcal{A}_i$. If $\lambda_i(x_i) = +\infty$, then for all $\beta \geq 0$, $x_i^* + \beta(x_i - x_i^*) \in S_i$. Since $O^+ S_i = \{0\}$, that implies $x_i = x_i^*$ which contradicts $x_i^* \in \mathcal{A}_i^c$.

Case 3. Indeed, let $(x_i, d_i) \in X_i \times R_+$. Take any $x_i' \in S_i$ and $d_i' > d_i$. We have
$$\hat{u}_i(x_i', d_i') = u_i(x_i') + \mu d_i' > u_i(x_i) + \mu d_i \geq \hat{u}_i(x_i, d_i).$$

We have proved that for any i, $\hat{u}_i$ has no satiation point.

Summing up, Assumptions $(H_1) - (H_4)$ are fulfilled in economy $\hat{\mathcal{E}}$. From Theorem 1, there exists a Walras quasi-equilibrium $((x_i^*, d_i^*)_{i \in I}, (p^*, q^*))$ with $(p^*, q^*) \neq (0, 0)$. It satisfies:

$$(i) \quad \sum_{i \in I} (x_i^*, d_i^*) = \sum_{i \in I} (e_i, \delta_i),$$

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(ii) for any $i \in I$, $p^* \cdot x_i^* + q^* d_i^* = p^* \cdot e_i + q^* \delta_i$.

Observe that the price q^* must be nonnegative.

We claim that $((x_i^*)_{i \in I}, p^*))$ is a quasi-equilibrium with dividends $(q^* \delta_i)_{i \in I}$. Indeed, first, we have

$$\forall i \in I, \ p^* \cdot x_i^* \le p^* \cdot e_i + q^* \delta_i.$$

Now, let $x_i \in X_i$ such that $u_i(x_i) > u_i(x_i^*)$. That implies $x_i^* \notin S_i$ and hence $\hat{u}_i(x_i^*, d_i^*) = u_i(x_i^*)$. We have $\hat{u}_i(x_i, 0) \ge u_i(x_i)$ and hence $\hat{u}_i(x_i, 0) > \hat{u}_i(x_i^*, d_i^*)$. Applying the previous theorem, we obtain

$$p^* x_i = p^* \cdot x_i + q^* \times 0 \ge p^* \cdot e_i + q^* \delta_i.$$

We have proved our proposition. $\square$

3.2 New nonsatiation assumption

The above concept of equilibrium with dividends is used in the literature whenever the standard nonsatiation assumption fails to be satisfied, that is to say, the satiation area intersects with individually feasible consumption set. The underlying idea is to allow the nonsatiated consumers to capture, through dividends, the budget surplus created by non budget-binding optimal consumptions of satiated consumers. A shortcoming of equilibrium with dividends is that, granting additional incomes to consumers could possibly be inconsistent with the spirit of decentralized markets.

In the following we introduce our new nonsatiation assumption. First, observe that under $(H_1) - (H_3)$, for every $i \in I$, there exists $\hat{x}_i \in \mathcal{A}_i$ which satisfies $u_i(\hat{x}_i) = M_i = \max\{u_i(x_i) \mid x_i \in \mathcal{A}_i\}$. for every $i \in I$.

Assumption (H_4) could be rewritten in another way using M_i:

(H_4) For every i, there exists $x_i' \in A_i^c$ such that $u_i(x_i') > M_i$.

We introduce a new nonsatiation condition (H_4') weaker than (H_4)

(H_4') For every i, there exists $x_i' \in \mathcal{A}_i^c$ such that $u_i(x_i') \ge M_i$.

Assumption (H_4') allows to have satiation points inside the individually feasible consumptions set, provided that the satiation area is not a subset of the individually feasible consumption set.

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We now state the main contribution of this paper. We demonstrate that using our new nonsatiation assumption (H'_4) leads us to the existence of a Walras quasi-equilibrium. Hence, we show that we show the concept of (quasi-)equilibrium with dividends is relevant only when the satiation area is a subset of individually feasible consumption set.

Theorem 2 *Assume* $(H_1), (H_2), (H'_4)$ *and the compactness of* $\mathcal{A}$. *Then there exists a Walras quasi-equilibrium.* [4]

Proof. For any i, from (H'_4), one can take some $x'_i \in \mathcal{A}^c_i$ such that $u_i(x'_i) \geq M_i$. Let S_i be the set of satiation points of agent i.

Let $(\delta_i > 0)_{i \in I}$. Let $B(0, a)$ denote the closed ball centered at the origin with radius a. Choose a sufficiently large such that for all i, both e_i and x'_i belong to int$B(0, a)$, and $\mathcal{A}_i \subset$ int$B(0, a)$. Consider the economy $\{\mathcal{E}^a\}$ defined by $\mathcal{E}^a = (X^a_i, u_i, e_i)_{i \in I}$ where $X^a_i = X_i \cap B(0, a)$. This economy $\mathcal{E}^a$ satisfies the assumptions of Proposition 1. Let $\hat{u}^a_i : X^a_i \times R_+ \to R$ be the modified utility associated with $\mathcal{E}^a$ as in the proof of Proposition 1. The utilities $\{(\hat{u}^a_i)_{i \in I}\}$ have no satiation points and are strictly quasi-concave. Hence, likewise Proposition 1, from Theorem 1, there exists a Walras quasi-equilibrium $((x^*_i, d^*_i)_{i \in I}, (p^*, q^*))$ with $(p^*, q^*) \neq (0, 0)$. It satisfies:

$$(i) \ \sum_{i \in I}(x^*_i, d^*_i) = \sum_{i \in I}(e_i, \delta_i),$$

$$(ii) \text{ for any } i \in I, \ p^* \cdot x^*_i + q^* d^*_i = p^* \cdot e_i + q^* \delta_i.$$

(iii) for any $i \in I$, if $\hat{u}^a_i(x_i, d_i) > \hat{u}^a_i(x^*_i, d^*_i)$ then $p^* \cdot x_i + q^* d_i \geq p^* \cdot e_i + q^* \delta_i$.

Now, we show that $q^* d^*_i = 0$. It is clear that from (H'_4) we have just to consider only cases 1 and 2 in Proposition 1. Thus, one has $\hat{u}^a_i(x'_i, 0) > \hat{u}^a_i(x^*_i, d^*_i) = \hat{u}^a_i(x^*_i, 0)$.
For any $\lambda \in]0, 1]$, from the strict quasi-concavity of $\hat{u}^a_i$, it follows that $\hat{u}^a_i(\lambda x'_i + (1-\lambda)x^*_i, 0) > \hat{u}^a_i(x^*_i, d^*_i) = \hat{u}^a_i(x^*_i, 0)$ and hence $p^* \cdot (\lambda x'_i + (1-\lambda)x^*_i) \geq p^* \cdot x^*_i + q^* d^*_i$. Letting λ converge to zero, we obtain $q^* d^*_i \leq 0$. Thus $q^* d^*_i = 0$

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[4]In a previous version, we assumed the compactness of U instead of the compactness of $\mathcal{A}$. The proof was not correct. Martins-da-Rocha and Monteiro [13] provide a counter example.

since q^* is nonnegative. Since $\sum_{i \in I} d_i^* = \sum_{i \in I} \delta_i > 0$, it follows that $q^* = 0$. In this case, one deduces $\|p^*\| = 1$ and therefore $((x_i^*)_{i \in I}, p^*)$ is a Walras quasi-equilibrium for $\mathcal{E}^a$.

We claim that $((x_i^*)_{i \in I}, p^*)$ is a Walras quasi-equilibrium for the initial economy. Indeed, we have:

$$\sum_{i \in I} x_i^* = \sum_{i \in I} e_i$$
$$\forall i, \; p^* \cdot x_i^* = p^* \cdot e_i.$$

Now, let $u_i(x_i) > u_i(x_i^*)$. For any $\lambda \in \,]0, 1]$, from the strict quasi-concavity of u_i, it follows that $u_i(\lambda x_i + (1 - \lambda)x_i^*) > u_i(x_i^*)$. Since $\mathcal{A}_i \subset \mathrm{int}B(0, a)$, for λ close to 0, $\lambda x_i + (1 - \lambda)x_i^* \in X_i^a$. Then,

$$\widehat{u}_i^a(\lambda x_i + (1 - \lambda)x_i^*) \geq u_i(\lambda x_i + (1 - \lambda)x_i^*) > u_i(x_i^*) = \widehat{u}_i^a(x_i^*),$$

and hence,

$$p^* \cdot (\lambda x_i + (1 - \lambda)x_i^*) \geq p^* \cdot e_i = p^* \cdot x_i^*,$$

which implies, $p^* \cdot x_i \geq p^* \cdot e_i$. □

4 Securities market

In securities markets with short-selling, Werner (1987) introduces a nonsatiation condition which requires each trader to have a useful portfolio. Accordingly, Werner (1987) proves the existence of a competitive equilibrium in securities markets. For each agent $i \in I$, we define the weakly preferred set at x_i $\widehat{P}_i(x_i) = \{x \in X_i \mid u_i(x) \geq u_i(x_i)\}$. Under assumptions $(H_1) - (H_2)$, the weak preferred set $\widehat{P}_i(x_i)$ is convex and closed for every $x_i \in X_i$. We define the ith *agent's arbitrage cone* at $x_i \in X_i$ as, $O^+\widehat{P}_i(x_i)$, the recession cone of the weakly preferred set $\widehat{P}_i(x_i)$. Also, we define the lineality set $L_i(x_i)$ as the largest subspace contained in the arbitrage cone $O^+\widehat{P}_i(x_i)$. For notational simplicity, we denote each agent's arbitrage cone and lineality space at endowments in a special way. In particular, we will let

$$R_i := O^+\widehat{P}_i(e_i), \text{ and } L_i := L(e_i).$$

Werner (1987) assumes the two following assumptions:

[W1] *Uniformity* $O^+ \widehat{P}_i(x_i) = R_i$, for all, $x_i \in X_i$, for each $i \in I$.

[W2] *Werner's nonsatiation* $R_i \setminus L_i \neq \emptyset$, for each $i \in I$.

The first assumption asserts that every agent has a uniform arbitrage cones. The second assumption is viewed as a nonsatiation assumption. It requires that there exists a *useful* net trade vector $r_i \in R_i \setminus L_i$. This is a portfolio which, when added, at any rate, to any given portfolio increases the trader's utility.

Werner (1987) also introduces a no-arbitrage condition [WNAC], which stipulates that there exists a price system at which the value of all useful portfolios is positive.

[WNAC] The economy $\mathcal{E}$ satisfies $\bigcap_{i=1}^{I} S_i^W \neq \emptyset$, where $S_i^W = \{p \in R^\ell \mid p \cdot y > 0, \forall y \in R_i \setminus L_i\}$ is Werner's cone of *no-arbitrage prices*.

Let $\mathcal{A}^\perp$ denote the orthogonal projection of $\mathcal{A}$ onto $\prod_{i \in I} L_i^\perp$. It follows from Allouch, Le Van, and Page (2002) that [WNAC] is equivalent to the compactness of $\mathcal{A}^\perp$ while [WNAC] and [W1] imply that U is compact. Hence, Assumption (H_3) is satisfied.

In the following proposition we show that Werner's nonsatiation implies assumption (H_4').

Proposition 2 *Assume* $(H_1)-(H_2)$, [W1] *and* [WNAC]. *Then,* [W2] *implies* (H_4').

Proof. Let $i \in I$. Since [WNAC] and [W1] imply that U is compact, there exists $x_i^* = \operatorname{argmax}_{\mathcal{A}_i} u_i(.)$ Let $r_i \in R_i \setminus L_i$. Write $x_i^* = y_i^* + z_i^*$, $r_i = s_i + t_i$ with $y_i^* \in \mathcal{A}_i^\perp$, $s_i \in L_i^\perp$, $z_i^* \in L_i$, $t_i \in L_i$. Since $r_i \notin L_i$, we have $s_i \neq 0$. Observe that [W1] implies $u_i(x_i) = u_i(x_i + t_i), \forall x_i \in X_i, \forall t_i \in L_i$. Therefore, we have

$$u_i(y_i^* + \lambda s_i) = u_i(y_i^* + \lambda r_i) \geq u_i(y_i^*) = u_i(x_i^*), \ \forall \lambda \geq 0,$$

and hence, for $\lambda > 0$, large enough, we must have $y_i^* + \lambda s_i \notin \mathcal{A}_i^\perp$ since $\mathcal{A}_i^\perp$ is compact. In particular $x_i^* + \lambda s_i \notin \mathcal{A}_i$, and $u_i(x_i^* + \lambda s_i) \geq u_i(x_i^*)$. $\square$

Proposition 3 *Assume* $(H_1) - (H_2)$, [W1]-[W2] *and* [WNAC]. *Then, there exists a Walras quasi-equilibrium.*

Proof. Consider, for any i, the direct decomposition

$$X_i = (X_i \cap L_i^{\perp}) + L_i.$$

Let π_i denote the orthogonal projection of x onto $L_i^{\perp}$. Observe that under [W1], we have for any i, $u_i(x_i) = u_i(y_i)$, where y_i is the projection of x_i onto $L_i^{\perp}$. In particular, $u_i(x + l_i) = u_i(x)$ for any $x \in X_i$, any $l_i \in L_i$.

Let S_i denote the set of satiation points of agent i and $S_i^{\perp} = S_i \cap L_i^{\perp}$. When S_i is non-empty, take $\widehat{x}_i \in S_i^{\perp} \cap \mathcal{A}_i^c$. Let $B(0, a)$ be a closed ball of R^l which is sufficiently large to contain in its interior $\mathcal{A}_i^{\perp}$, $\widehat{x}_i$ and e_i for every i. Let

$$X_i^a = \left[X_i \cap L_i^{\perp} \cap B(0, a) \right] + L_i.$$

Define on $X_i^a \cap S_i$ the function μ_i^a as follows:

$$x_i \in S_i \cap X_i^a \longrightarrow \mu_i^a(x_i) = \sup\{\beta \geq 0 : \widehat{x}_i + \beta(\pi_i(x_i) - \widehat{x}_i) \in S_i \cap X_i^a\}$$

One can check that μ_i^a is upper semi-continuous, strictly quasi-concave. Moreover, $\mu_i^a(x_i + t_i) = \mu_i^a(x_i), \forall x_i \in X_i^a, \forall t_i \in L_i$ and $\mu_i^a(x_i) = +\infty$ iff $\pi_i(x_i) = \widehat{x}_i$. Let

$$
\begin{aligned}
u_i^a(x_i) &= u_i(x_i) + 1 - \frac{1}{\mu_i^a(x_i)}, \text{ if } x_i \in X_i^a \cap S_i \\
u_i^a(x_i) &= u_i(x_i) \text{ otherwise.}
\end{aligned}
$$

Note that $u_i^a(x_i + t_i) = u_i^a(x_i)$ for any $x_i \in X_i^a$, any $t_i \in L_i$. Consider economy $\mathcal{E}^a = \{(X_i^a, u_i^a, e_i)_i\}$. Let $\mathcal{A}^a$ be the individually rational feasible allocations set of this economy, and $\mathcal{A}_i^a$ be its projection onto X_i^a. It is easy to check that u_i^a is upper semi-continuous, strictly quasi-concave and it has no satiation point on $\mathcal{A}_i^a$. Assumptions [WNAC] and [W1] imply the compactness of the individually rational utility set of $\mathcal{E}^a$. From Theorem 1, $\mathcal{E}^a$ admits a quasi-equilibrium $((x_i^*)_i, p^*)$. We claim that it also is a quasi-equilibrium for the initial economy. Indeed, we have

$$
\begin{aligned}
\sum_{i \in I} x_i^* &= \sum_{i \in I} e_i \\
\forall i, \ p^* \cdot x_i^* &= p^* \cdot e_i
\end{aligned}
$$

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Now, let $u_i(x_i) > u_i(x_i^*)$. In this case, x_i^* is not in S_i. For any $\lambda \in (0,1)$ one has $u_i(\lambda x_i + (1-\lambda)x_i^*) > u_i(x_i^*)$. For λ close to zero, $\lambda x_i + (1-\lambda)x_i^* \in X_i^a$ and thus,

$$u_i^a(\lambda x_i + (1-\lambda)x_i^*) \geq u_i(\lambda x_i + (1-\lambda)x_i^*) > u_i(x_i^*) = u_i^a(x_i^*).$$

Hence, $p^* \cdot (\lambda x_i + (1-\lambda)x_i^*) \geq p^* \cdot e_i = p^* \cdot x_i^*$ which gives $p^* \cdot x_i \geq p^* \cdot e_i$. $\square$

4.1 Example

Now, we provide an example where both the standard nonsatiation assumption (H_4) and Werner's nonsatiation [W2] fail to be satisfied. However, our new nonsatiation assumption (H_4') holds. In this example, we have the existence of a competitive equilibrium that could not have been inferred from standard existence theorems.

Example Consider the economy with two consumers and two commodities. Consumer 1 has the following characteristics: $X_1 = [0,10] \times [0,10]$,

$$u_1(x_1, y_1) = \begin{cases} \min\{x_1, y_1\}, & \text{if either } x_1 \in [0,3] \text{ or } y_1 \in [0,3], \\ 3 & \text{otherwise.} \end{cases}$$

$e_1 = (6,2)$.

Consumer 2 has the following characteristics: $X_2 = R_+^2$ $u_2(x_2, y_2) = x_2 + y_2$
$e_2 = (2,6)$.
We have $u_1(e_1) = 2, u_2(e_2) = 8$.
The satiation set of agent 1 is $S_1 = [3,10] \times [3,10]$. Let $\zeta_1 = (x_1, y_1)$, $\zeta_2 = (x_2, y_2)$. The set of individually rational feasible allocations is:

$$A = \{(\zeta_1, \zeta_2) \in X_1 \times X_2 : \zeta_1 + \zeta_2 = (8,8), \text{ and } u_1(\zeta_1) \geq 2, u_2(\zeta_2) \geq 8\}.$$

It is easy to see that $(3,3) \in \mathcal{A}_1$ and hence $M_1 = 3$. But observe that $(10,10) \notin \mathcal{A}_1$ and satisfies $u_1(10,10) = 3 = M_1$. In other words, for agent 1, Assumption (H_4) is not satisfied. It is also worth noticing that Werner's nonsatiation is not satisfied by agent 1 since X_1 is a compact set, and therefore $R_1 = \{0\}$. However, it is obvious that Assumption (H_4') is satisfied by both consumers.
One can easily show that the allocation $((4,4),(4,4))$ together with the price $(1,1)$ is an equilibrium for the economy.

5 Appendix

Lemma 1 *Let B be a closed, convex set of R^n and x^* be in B. Let $\lambda : B \to R_+ \cup \{+\infty\}$ be defined by*

$$\lambda(x) = \sup\{\beta \geq 0 : x^* + \beta(x - x^*) \in B\}.$$

Then λ is upper semicontinuous and strictly quasi-concave. Proof. First,

observe that $\lambda(x) \geq 1, \forall x \in B$.

(i) *Upper semicontinuity of λ:* First we have:

$$x - x^* \in O^+ B \iff \lambda(x) = +\infty.$$

Let $x^n \to x$. If $\limsup \lambda(x^n) = +\infty$, take a subsequence which satisfies $\lim \lambda(x^\nu) = +\infty$. Let $\varepsilon > 0$ be any small positive number. We have

$$x^* + (\lambda(x^\nu) - \varepsilon)(x^\nu - x^*) = z^\nu \in B, \ \forall \nu,$$

or

$$(x^\nu - x^*) = \frac{z^\nu - x^*}{\lambda(x^\nu) - \varepsilon}.$$

Let $\nu \to +\infty$. We have $\frac{z^\nu - x^*}{\lambda(x^\nu) - \varepsilon} \to x - x^*$, hence, $x - x^* \in O^+ B$. That implies $\lambda(x) = +\infty$ and $\limsup \lambda(x^n) = \lambda(x)$.
Now, assume $\limsup \lambda(x^n) = A < +\infty$. Without loss of generality, one can assume that $\lambda(x^n) \to A$. Take any $\varepsilon > 0$ small enough. Then, $\forall n, x^* + (\lambda(x^n) - \varepsilon)(x^n - x^*) \in B$. Let n go to infinity. Then $x^* + (A - \varepsilon)(x - x^*) \in B$. That implies $\lambda(x) \geq A - \varepsilon$ for any $\varepsilon > 0$ small enough. In other words, $\lambda(x) \geq \limsup \lambda(x^n)$. We have proved that λ is upper semicontinuous.

(ii) *Quasi-concavity of λ:* Let $x_1 \in B$, $x_2 \in B$, $\theta \in (0, 1)$ and $x = \theta x_1 + (1 - \theta)x_2$. Assume $\lambda(x_1) \leq \lambda(x_2)$. As before take any $\varepsilon > 0$ small enough. We then have

$$x^* + (\lambda(x_1) - \varepsilon)(x_1 - x^*) \in B,$$

and

$$x^* + (\lambda(x_1) - \varepsilon)(x_2 - x^*) \in B,$$

since $\lambda(x_1) \leq \lambda(x_2)$. Thus

$$\theta(x^* + (\lambda(x_1) - \varepsilon)(x_1 - x^*)) + (1 - \theta)(x^* + (\lambda(x_1) - \varepsilon)(x_2 - x^*)) \in B$$

since B is convex. We obtain $x^* + (\lambda(x_1) - \varepsilon)(\theta x_1 + (1-\theta)x_2 - x^*) \in B$. Hence $\lambda(\theta x_1 + (1-\theta)x_2) \geq \lambda(x_1) - \varepsilon$ for any $\varepsilon > 0$ small enough. In other words, $\lambda(\theta x_1 + (1-\theta)x_2) \geq \min\{\lambda(x_1), \lambda(x_2)\}$. We have proved the quasi-concavity of λ.

(iii) *Strict quasi-concavity of* λ: Let $\lambda(x_2) > \lambda(x_1)$. We first claim that $x^* + \lambda(x_1)(x_1 - x^*) \in B$. Indeed, $\forall n, x^* + (\lambda(x_1) - \frac{1}{n})(x_1 - x^*) \in B$. Let $n \to +\infty$. The closedness of B implies that $x^* + \lambda(x_1)(x_1 - x^*) \in B$.

Now, let $\theta \in]0, 1[$. We claim that $\lambda(\theta x_1 + (1-\theta)x_2) > \lambda(x_1)$. For that, for short, write $\lambda_1 = \lambda(x_1), \lambda_2 = \lambda(x_2)$. Let A satisfy $\lambda_1 < A < \lambda_2$. Then $x^* + A(x_2 - x^*) \in B$. Hence

$$\frac{\theta A}{\lambda_1(1-\theta) + \theta A}(x^* + \lambda_1(x_1 - x^*)) + \frac{\lambda_1(1-\theta)}{\lambda_1(1-\theta) + \theta A}(x^* + A(x_2 - x^*)) \in B,$$

or equivalently

$$x^* + \frac{A\lambda_1}{\lambda_1(1-\theta) + \theta A}(\theta x_1 + (1-\theta)x_2 - x^*) \in B.$$

Thus, $\lambda(\theta x_1 + (1-\theta)x_2) \geq \frac{A\lambda_1}{\lambda_1(1-\theta) + \theta A} > \lambda_1$, since $A > \lambda_1$. $\square$

Claim 1 *The utility function* $\widehat{u}_i$ *is strictly quasi-concave.*
Proof. Let

$$\widehat{u}_i(x_2, d_2) > \widehat{u}_i(x_1, d_1) \tag{1}$$

and $\theta \in]0, 1[$. We claim that

$$\widehat{u}_i(\theta(x_2, d_2) + (1-\theta)(x_1, d_1)) > \widehat{u}_i(x_1, d_1). \tag{2}$$

Let us distinguish the three cases again:

Case 1. The claim is obviously true since (1) is equivalent to $u_i(x_2) > u_i(x_1)$ and u_i is assumed to be strictly quasi-concave.

Case 2. We have two sub-cases.
(a) If $x_1 \in S_i$, then $x_2 \in S_i$. In this case, (1) is equivalent to $\lambda_i(x_2) > \lambda_i(x_1)$. From Lemma 1, λ_i is strictly quasi-concave. Hence (2) is true, since S_i is convex, and therefore $\theta x_2 + (1-\theta)x_1 \in S_i$.

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(b) If $x_1 \notin S_i$. Then (1) is equivalent to $u_i(x_2) > u_i(x_1)$. Since u_i is a strictly quasi-concave function, we obtain

$$u_i((1-\theta)x_1 + \theta x_2) > u_i(x_1).$$

Therefore, one has

$$\widehat{u}_i(\theta(x_2, d_2) + (1-\theta)(x_1, d_1)) \geq u_i(\theta x_2 + (1-\theta)x_1) > u_i(x_1) = \widehat{u}_i(x_1, d_1),$$

and consequently (2) is true.

Case 3. We can consider the following cases: (a)If $x_1 \in S_i$, then $x_2 \in S_i$. Hence, we have

$$\widehat{u}_i(x_1, d_1) = M_i + \mu d_1, \ \widehat{u}_i(x_2, d_2) = M_i + \mu d_2.$$

It follows from (1) that $d_2 > d_1$. Hence,

$$(1-\theta)d_1 + \theta d_2 > (1-\theta)d_1 + \theta d_1 = d_1.$$

Since $(1-\theta)x_1 + \theta x_2 \in S_i$, we deduce

$$\widehat{u}_i((1-\theta)x_1 + \theta x_2, (1-\theta)d_1 + \theta d_2) =$$

$$M_i + \mu((1-\theta)d_1 + \theta d_2) > M_i + \mu d_1 = \widehat{u}_i(x_1, d_1).$$

(b)If $x_1 \notin S_i$. Then (1) implies $u_i(x_2) > u_i(x_1)$. Since u_i is a strictly quasi-concave function, we obtain

$$u_i((1-\theta)x_1 + \theta x_2) > u_i(x_1).$$

Then, it follows that

$$\widehat{u}_i((1-\theta)x_1 + \theta x_2, (1-\theta)d_1 + \theta d_2) \geq u_i((1-\theta)x_1 + \theta x_2) > u_i(x_1) = \widehat{u}_i(x_1, d_1).$$

The proof of the claim is complete.$\square$

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