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# Growth and convergence in a model with renewable and non-renewable resources: existence, transitional dynamics, and empirical evidence

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# Growth and convergence in a model with renewable and non-renewable resources: existence, transitional dynamics, and empirical evidence *

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**Abstract:** This paper studies an optimal endogenous growth model using physical capital, labor and two kinds of natural resources in the final goods sector and employing labor to accumulate knowledge. Based on results in calculus of variations, a direct proof of existence of optimal solution is provided. Analytical solutions for the planner case and the balanced growth paths are found for a specific CRRA utility and Cobb-Douglas production function. Transitional dynamics to the steady state from the theoretical model are used to derive three convergence equations of output intensity growth rate, exhaustible resource growth rate and renewable growth rate, which are tested based on data on production and energy consumption in 27 OECD countries.

**Keywords:** Optimal growth, existence of equilibrium, transitional dynamics, energy, renewable resource, non-renewable resource.

**JEL Classification:** C61, D51, E13.

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# 1 Introduction

An important question that has captivated the attention of environmental economists is whether growth is sustainable in the presence of natural resource scarcity. The issue concerns both academics and public decision-makers, notably in the current context of increasing energy demands and the depletion of fossil fuels expected in the near future. The new growth theory gives the answer that with some technological properties, growth may be sustained in the long-run even if resource stock is finite. This conclusion can be successfully summarized in Smulders (2005): '*...that a society willing to spend enough on R&D can realize a steady state of technological change sufficient to offset the diminishing returns from capital-resource substitution and sustain long-run growth*', and in Bretschger (2005): '*...technological change has the potential to compensate for natural resource scarcity, diminishing returns to capital, poor input substitution, and material balance restrictions, but is limited by various restrictions like fading returns to innovative investments and rising research costs.*'

However, the literature does not pay enough attention to the existence of the proposed theoretical solution and their empirical justifications. This is particularly due to the difficulty of building a testable version from a complex theoretical framework.

A recent strand of literature concerns the search for empirical evidence for the proposed theoretical models. Pioneer works were done by Brock and Taylor (2004), Alvarez et al. (2005), Bretschger (2006), Miketa and Mulder (2005), and Mulder and De Groot (2007). Brock and Taylor (2004) proposed a Solow growth model with pollution together with empirical justification. They found evidence of an environmental Kuznets curve (an inverted-U shaped relationship between emissions and income) for OECD countries. They also showed that pollution emissions have a convergence feature, like income convergence in empirical growth studies. Alvarez et al. (2005) provided a Ramsey growth model with pollution which is compatible with the empirical finding about pollution convergence in a panel of European countries. Bretschger (2006) provided an empirical validation of the balanced growth path derived from an endogenous growth model with energy. The author also obtained that rising energy prices are not a threat to economic development. Miketa and Mulder (2005) and Mulder and De Groot (2007) found evidence for conditional convergence in energy productivity, i.e. convergence based on country-specific conditions, which seems support the underlying Solow-Swan growth model.

In line with this strand of literature, our paper addresses an endogenous growth model of which the results may be tested with real data. We provide a framework where technological change is endogenized and the production employs labor, physical capital, and both type of renewable and non-renewable resources.[1] The roles of renewable and non-renewable resources have been simultaneously analyzed in some recent studies, e.g. Tahvonen and Salo (2001), Gerlagh and van der Zwaan (2003), Tsur and Zemel (2003), André and Cerdá (2006), Grimaud and Rougé (2004, 2005, 2008), Growiec and Schumacher (2008). However, they are mainly related to the substitutability of natural resources (substitution between man-made and natural capitals, substitution between renewable and non-renewable resources) or technological conditions that ensure a sustainable economic growth, and are not explicitly concerned with empirical testing. Our paper investigates a different and more technical issue. We present a rigorous proof of the existence of the optimal solution of a general model, which is often assumed in the many papers in the literature. As always, the arguments for existence of solutions rely on compactness of feasible set and some form of continuity of objective function. We first prove the uniformly boundedness of feasible set (assumptions in d'Albis et al, 2002) that deduces the Lebesgue uniformly integrability. The theorem of Dunford-Pettis ( Dunford-Schwartz (1967)) which characterizes the Lebesgue uniformly integrability and the relatively weak compactness of feasible set is needed in the proof. Then we prove the set of feasible consumption paths is compact. Combined with compactness, upper semi-continuous of objective function is all that is necessary for existence of a maximum. For the proof we also refer the reader to Mazur's Lemma and Fatou's Lemma. Next, we propose an analytical model with explicit computation of the transitional dynamics and the balanced growth path. Finally, data on production and energy consumption of OECD countries are used to perform an empirical test based on the transitional equations of this analytical model.

The paper is organized as follows. After the Introduction, in Section 2 we introduce the general optimal endogenous growth model. The existence and uniqueness of a solution is shown in Section 3. A specific model is discussed in Sections 4 where the balanced growth path and analytical optimal growth

[1] Most of existing studies focus separately renewable and non-renewable resources. However, the literature is very abundant to be cited here. See Kolstad and Krautkraemer (1993), Barbrier (1999), Scholz and Ziemes (1999), Bretschger (2005), Smulders (2005), and Brock and Taylor (2006), among others, for some literature overviews.

rates are found. Section 5 presents the empirical test of the analytical model based on the results of transitional dynamics. Section 6 concludes.

# 2 The model

The model can be heuristically described as follows. The aggregate output produced from the labor, physical capital and two types of natural resources: non-renewable resources (e.g., fossil fuels) and renewable resources (solar, thermal, biomass, etc.). The final product is shared between consumption and investment in physical capital. The representative consumer derives her utility from consumption. The production function takes the form

$$Y = A^{\theta} f(K, L_Y, Q, R)$$

where $A$, $L_Y, K, Q$ and $R$ represent the technological level, labor input, physical capital input, non-renewable resource (or fossil energy), and renewable resource (or non-fossil energy), respectively.

We assume the law of motion of technological change is

$$\dot{A} = \Psi(A, L_A)$$

where $L_A$ is labor employed for research and $\Psi$ is a knowledge production function. Normalizing the total flow of labor we have

$$L_Y + L_A = 1.$$

The final output can be allocated between consumption and investment (or capital accumulation)

$$\dot{K} = Y - C - \delta K = I - \delta K,$$

where $\delta \in (0, 1)$ is the depreciate rate of the stock of capital.

It is standard that the dynamics of stock non-renewable resource is following

$$\dot{S}_{Q_t} = -Q_t,$$

where $S_{Q_t}$ is the stock of exhaustible resource at time $t$. It follows from this equation and non-negative restriction on $Q$ that

$$\int_0^{\infty} Q dt \leq S_{Q_0}.$$

The dynamics of stocks of renewable is

$$\dot{S}_{R_t} = h(S_{R_t}) - R_t$$

where $h$ is a regeneration function.

The representative consumer's utility function is given by

$$U = \int_0^{\infty} u(C_t) e^{-\rho t} dt$$

From now on, as it is not necessary, the time index is not included for simplifying our notation.

# 3 Existence of optimal solution

In this section, we prove the existence of solution to the social problem (P):

$$\max \int_0^{\infty} u(C) e^{-\rho t} dt$$

subject to

$$\begin{aligned}
\dot{S}_R &= h(S_R) - R, && (1)\\
\dot{S}_Q &= -Q, && (2)\\
\dot{A} &= \Psi(A, L_A), && (3)\\
\dot{K} &= A^{\theta} f(K, L_Y, Q, R) - C - \delta K, && (4)\\
L_A + L_Y &= 1, && (5)
\end{aligned}$$

and $C \geq 0, K \geq 0, A \geq 0, 0 \leq L_Y \leq 1, 0 \leq L_A \leq 1$, given $A_0, L_{Y0}, K_0, S_{Q_0}, S_{R_0}$.

Note that $\theta$ may be greater than 1, the maximal Hamiltonian is not concave in every state variable so the Arrow or Mangasarian sufficiency theorem does not apply in our model. In such an endogenous natural resources dynamic model with non-concave maximal Hamitonian, Kuhn-Tucker first-order conditions together with transversality conditions are necessary and sufficient conditions for an optimal solution is still a conjecture. (see Groth and Schou (2007, footnote 26, p. 93) or Groth and Schou (2002)). As always, the arguments for existence of solutions rely on compactness of feasible set and some form of continuity of objective function. We first prove the uniformly boundedness of feasible set (assumptions in d'Albis et al, 2002) that deduces the Lebesgue uniformly integrability. The theorem of Dunford-Pettis ( Dunford-Schwartz (1967)) which characterizes the Lebesgue uniformly integrability and the relatively weak compactness of feasible set is needed in the proof. Then we prove the set of feasible consumption paths is compact. Combined with compactness, upper semi-continuous of objective function is all that is necessary for existence of a maximum. For the proof we refer the reader to Dunford-Pettis's Theorem, Mazur's Lemma and Fatou's Lemma in the Appendix.

Let us denote by $L^1(e^{-\rho t})$ is the set of function $f$ verifying $\int_0^\infty |f(t)| e^{-\rho t} dt < \infty$. Recall that $f_i(t) \in L^1(e^{-\rho t})$ weakly converges to $f(t) \in L^1(e^{-\rho t})$ for the topology $\sigma(L^1(e^{-\rho t}), L^\infty)$ (written as $f_i \rightharpoonup f$ ) if and only if for every $\Psi \in L^\infty$, $\int_0^\infty f_i q e^{-\rho t} dt$ converges to $\int_0^\infty f \Psi e^{-\rho t} dt$ as $i \to \infty$. ( written as $\int_0^\infty f_i \Psi e^{-\rho t} dt \longrightarrow \int_0^\infty f \Psi e^{-\rho t} dt$ ).

When writing $f_i \longrightarrow f^*$ we mean that for every $t \in [0, \infty), \lim_{i \to \infty} f_i(t) = f^*(t)$.

We make the following assumptions:

H1. *The function $u(C) : R_+ \to R$ is strictly concave, increasing and continuous.*

H2. *Functions $f(K, L_Y, Q, R) : R_+^4 \to R_+$ is continuously differentiable, increasing on all arguments and*

$$\begin{aligned} f(0,.) &= 0, \\ \lim_{K \to +\infty} f_K(K, 1, S_{Q_0}, S_{R_0}) &\leq 0. \end{aligned}$$

H3. *Functions $\Psi(A, L_A) : R_+^2 \to R_+$ is continuously differentiable, increasing in both arguments.Moreover, there exists a constant $b$ such that*

$$\Psi(A, L_A) \leq bA.$$

H4. *Functions $h(S_R) : R_+ \to R_+$ is continuously differentiable increasing and there exists a constant $m$ such that $h(S_R) \leq m S_R$.*

H5. *There exists $\kappa \geq 0, \kappa \neq \infty, \kappa \geq 0, \kappa \neq \infty$* such that $-\kappa \leq \dot{K}/K$ and $-\mu \leq \dot{S}_R/S_R, -\pi \leq \dot{S}_Q/S_Q$.

H6. $\rho > \max\{b, m, b\theta\}$.

H1-H4 are standards but we do not require the concavity of any function in the technology. Assuming $\Psi(A, L_A) \leq bA$ has been used in Chichilnisky (1981) and it is weaker than the standard assumption $\lim_{A \to \infty} \Psi_A = 0$ ($\Psi_A = \frac{\partial \Psi(A, L_A)}{\partial A}$) and means that after certain levels of technical change the technology is constrained in its knowledge capital increases of productivity by the costs of maintenance, represented by the depreciation parameter $b$. Similarly for assumption on regeneration function $h$.

Assumption H5 is reasonable. It implies that it is not possible that the growth rate of physical capital or stock of renewable resource converges to $-\infty$ rapidly and is weaker than those used in the literature where $\kappa$ is a physical depreciation rate (Chichilnisky (1981), d'Albis et al (2008)). Let us define the net investment : $I = \dot{K} - \delta K = A^\theta f(K, L_Y, Q, R) - C$. Then H5 implies there exist $\kappa \geq 0, \kappa \neq \infty$ such that $I + (\kappa - \delta)K \geq 0$. Thus if the standard assumption of non-negative investment holds (that means capital goods cannot be converted back into consumption goods) then H5 holds with $\kappa = \delta$. Therefore assumption non-negative investment is stronger than A.6 ($\kappa$ can take any value except for infinity). H4 is similar to A4 in d'Albis et al (2008) which ensures a finite value of objective function and the maximal growth rate of the output is less than discount rate.

**Lemma 1** *Let us denote by $\mathcal{K} = (L_A, L_Y, Q, R, S_Q, S_R, A, K, C)$ the feasible path from $A_0, L_{Y0}, K_0, S_{Q_0}, S_{R_0}$ which satisfies (1)-(5) and $C \geq 0, K \geq 0, A \geq 0, Q \geq 0, R \geq 0, 0 \leq L_Y \leq 1, 0 \leq L_A \leq 1$. Then $\mathcal{K}$ is relatively weak compact in $L^1(e^{-\rho t})$.*

**Proof.** By (1) and assumption H4 we have $\dot{S}_R \leq h(S_R) \leq mS_R$ and we get $\dot{S}_R/S_R \leq m$. Thus, there exists $\overline{S}$ such that

$$\begin{aligned} 0 &\leq S_R \leq \overline{S}e^{mt}, \\ \dot{S}_R &\leq m\overline{S}e^{mt}. \end{aligned} \tag{6}$$

Thus, $S_R$ belongs to the space $L^1(e^{-\rho t})$ since

$$0 \leq \int_0^\infty S_R e^{-\rho t}dt \leq \overline{S}\int_0^\infty e^{(m-\rho)t}dt < +\infty.$$

According to H4, $-\dot{S}_R \leq \mu S_R \leq \mu\overline{S}e^{mt}$. It follows from (6) that $\left|\dot{S}_R\right| \leq \max\{m\overline{S}, \mu\overline{S}\}e^{mt}$ and

$$\int_0^\infty \left|\dot{S}_R\right| e^{-\rho t}dt \leq \max\{m\overline{S}, \mu\overline{S}\}\int_0^\infty e^{(m-\rho)t}dt < +\infty.$$

Since $0 \leq R = h(S_R) - \dot{S}_R \leq (m+\mu)\overline{S}e^{mt}$. Therefore we have

$$0 \leq \int_0^\infty Re^{-\rho t}dt \leq (m+\mu)\overline{S}\int_0^\infty e^{(m-\rho)t}dt < +\infty.$$

It follows from (2) that $0 \leq \int_0^t Q_s ds = -\int_0^t \dot{S}_{Q_s}ds = S_{Q_0} - S_Q \leq S_{Q_0}$. Thus $\left|\dot{S}_Q\right| = Q$, $S_Q \leq S_{Q_0}$ and $Q = -\dot{S}_Q \leq \pi S_Q$. We then have

$$\begin{aligned} \int_0^\infty S_Q e^{-\rho t}dt &\leq S_{Q_0}\int_0^\infty e^{-\rho t}dt < +\infty \\ \int_0^\infty Qe^{-\rho t}dt &= \int_0^\infty \left|\dot{S}_Q\right| e^{-\rho t}dt \leq \pi S_{Q_0}\int_0^\infty e^{-\rho t}dt < +\infty. \end{aligned}$$

Since $\lim_{K\to+\infty} f_K(K,1,S_{Q_0},S_{R_0}) \leq 0$, for any $\zeta \in (0, \rho - b\theta)$ there exist a constant $B_0$ such that

$$f(K, L_Y, Q, R) \leq B_0 + \zeta K.$$

It follows that

$$\dot{K} \leq B_0 + \zeta K.$$

Multiply by $e^{-\zeta s}$ we get $e^{-\zeta s}\dot{K} - \zeta K e^{-\zeta s} \leq B_0 e^{-\zeta s}$. Then we get

$$e^{-\zeta t}K = \int_0^t \frac{\partial(e^{-\zeta s}K)}{\partial s}ds + K_0 \leq \int_0^t B_0 e^{-\zeta s}ds + K_0 = \frac{-B_0 e^{-\zeta t}}{\zeta} + \frac{B_0 + \zeta K_0}{\zeta}.$$

This implies that there exists constant $B_1$ such that $K \leq B_1 e^{\zeta t}$. Hence $\int_0^\infty Ke^{-\rho t}dt \leq \int_0^\infty B_1 e^{(\zeta-\rho)t}dt < +\infty$.

Furthermore, since $-\dot{K} \leq \kappa K$ and $\dot{K} \leq B_0 + \zeta K \leq B_0 + \zeta B_1 e^{\zeta t}$, there exist a constant $B_2$ such that $\left|\dot{K}\right| \leq B_2 e^{\zeta t}$. Thus,

$$\int_0^\infty \left|\dot{K}\right| e^{-\rho t}dt \leq \int_0^\infty B_2 e^{(\zeta-\rho)t}dt < +\infty.$$

Since $\Psi(A, L_A) \leq bA$, we have $\dot{A}/A \leq b$. There exists a constant $D_1$ such that $A \leq D_1 e^{bt}$. Moreover, we have $0 \leq \dot{A} \leq bA \leq D_1 e^{bt}$. Therefore, $A$, $\left|\dot{A}\right|$ belong to $L^1(e^{-\rho t})$ because

$$\begin{aligned} 0 &\leq \int_0^\infty Ae^{-\rho t}dt \leq D_1\int_0^\infty e^{(b-\rho)t}dt < +\infty \\ 0 &\leq \int_0^\infty \left|\dot{A}\right| e^{-\rho t}dt \leq D_1\int_0^\infty e^{(b-\rho)t}dt < +\infty. \end{aligned}$$

As $-\dot{K} \leq \kappa K$, we have

$$\begin{aligned}
C &\leq A^{\theta} f(K, L_Y, Q, R) + (\kappa - \delta) K \\
&\leq {D_1}^{\theta} e^{b\theta t}(B + \zeta K) + (\kappa - \delta) B_1 e^{\zeta t} \\
&\leq {D_1}^{\theta} e^{b\theta t}(B + \zeta B_1 e^{\zeta t}) + (\kappa - \delta) B_1 e^{\zeta t} \\
&= {D_1}^{\theta} \zeta B_1 e^{(b\theta + \zeta)t} + {D_1}^{\theta} B e^{b\theta t} + (\kappa - \delta) B_1 e^{\zeta t}.
\end{aligned}$$

Thus, we can choose a positive constant $D_2 \geq {D_1}^{\theta} B + {D_1}^{\theta} \zeta B_1 + (\kappa - \delta) B_1$. Then

$$C \leq D_2 e^{(b\theta + \zeta)t} < D_2 e^{\rho t},$$

which implies

$$0 \leq \int_0^{\infty} C e^{-\rho t} dt < +\infty.$$

We have proven that $\mathcal{K}$ is uniformly bounded on $L^1(e^{-\rho t})$.

Moreover, $\lim_{a \to \infty} \int_a^{\infty} K e^{-\rho t} dt \leq \lim_{a \to \infty} \int_a^{\infty} B_1 e^{(\zeta - \rho)t} dt = 0$. This property is true for other variables in $\mathcal{K}$. Therefore $\mathcal{K}$ satisfies Dunford-Pettis theorem and it is relatively compact in the weak topology $\sigma(L^1(e^{-\rho t}), L^{\infty})$. ■

Since $\mathcal{K}$ is relatively compact in the weak topology $\sigma(L^1(e^{-\rho t}), L^{\infty})$, a sequence $X_i$ in $\mathcal{K}$ has convergent subsequences (denote by $X_i$ for simplicity of notation) which weakly converge to limit points in $L^1(e^{-\rho t})$.

The following Lemma shows that any weakly convergent sequence of control variables in $\mathcal{K}$ has a sequence of convex combinations of its members that converges pointwise to the same limit while the limit of weak convergence coincide with limit of pointwise convergence for state variables.

**Lemma 2** *i)Let $(K, A, S_R, S_Q)_i$ in $\mathcal{K}$ and suppose that $(K, A, S_R, S_Q)_i \rightharpoonup (K^*, A^*, S_R^*, S_Q^*)$.*

*Then $(K, A, S_R, S_Q)_i \to (K^*, A^*, S_R^*, S_Q^*)$ as $i \to \infty$ and $(\dot{K}, \dot{A}, \dot{S}_R, \dot{S}_Q)_i \rightharpoonup (\dot{K}^*, \dot{A}^*, \dot{S}_R^*, \dot{S}_Q^*)$ for the the topology $\sigma(L^1(e^{-\rho t}), L^{\infty})$.*

*ii) In addition, suppose that $\mathbf{Z}_i = (R, Q, \dot{K}, \dot{A}, \dot{S}_R, \dot{S}_Q)_i$ in $\mathcal{K}$ and $\mathbf{Z}_i \rightharpoonup \mathbf{Z}^* = (R^*, Q^*, \dot{K}^*, \dot{A}^*, \dot{S}_R^*, \dot{S}_Q^*)$ in $L^1(e^{-\rho t})$ then there exists a sequence of sets of real numbers $\{\omega_{i(n)} \mid i = n, ..., \mathcal{N}(n)\}$ such that $\omega_{i(n)} \geq 0$ and $\sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} = 1$ such that the sequence $(v_n)_{n \in N}$ defined by the convex combination $v_n = \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} \mathbf{Z}_i$ converges pointwise to $\mathbf{Z}^*$ as $n \to \infty$, i.e., for every $t \in [0, \infty), \lim_{n \to \infty} v_n(t) = \mathbf{Z}^*(t)$. It is clearly that, since $(L_A, L_Y,)_{i(n)} \in [0, 1]$, $(L_A, L_Y,)_{i(n)} \to (L_A^*, L_Y^*)$ as $n \to \infty$.*

**Proof.** For any $X_i \in \mathcal{K}$. We first claim that, for $t \in [0, \infty)$, $\int_0^t X_i dt \to \int_0^t X^* dt$. Note that $X_i \rightharpoonup X^*$ for the topology $\sigma(L^1(e^{-\rho t}), L^{\infty})$ if and only if for every $Y \in L^{\infty}$, $\int_0^{\infty} X_i Y e^{-\rho t} dt \to \int_0^{\infty} X^* Y e^{-\rho t} dt$.

Pick any $t$ in $[0, \infty)$ and let

$$Y(s) = \begin{cases} \frac{1}{e^{-\rho t}} \text{ if } s \in [0, t] \\ 0 \text{ if } s > t. \end{cases}$$

Therefore $Y(s) \in L^{\infty}$ and we get $\int_0^t X_i(s) ds = \int_0^{\infty} X_i(s) Y(s) e^{-\theta s} ds \to \int_0^{\infty} X^*(s) Y(s) e^{-\theta s} ds = \int_0^t X^*(s) ds$ .

Given that $K_i \rightharpoonup K^*$ and $\dot{K}_i \rightharpoonup y$ weakly in $L^1(e^{-\rho t})$, by the claim above, for all $t \in [0, \infty)$ we have $\int_0^t K_i ds \to \int_0^t y ds$ . This implies, for a fix $t$, $K_i \to \int_0^t y ds + K_0$. Thus $\int_0^t y ds + k_0 = K^*$. Therefore $\dot{K}^* = y$ or $\dot{K}_i \rightharpoonup \dot{K}^*$. The same reasoning applies for $(A, S_R, S_Q)_i$ in $\mathcal{K}$.

ii) A direct application of Mazur's Lemma. ■

We are now able to prove the existence of solution to the to the social planner's problem.

**Theorem 1** *Under Assumptions H.1-H.7, there exists a solution to the social planner's problem.*

**Proof.** Since $u$ is concave, for any $\bar{c} > 0$, $u(C) - u(\bar{c}) \leq u'(\bar{c})(C - \bar{c})$. Thus, if $C \in L^1(e^{-\rho t})$ then $\int_0^{\infty} u(C) e^{-\rho t} dt$ is well defined because

$$\int_0^{\infty} u(C) e^{-\rho t} dt \leq \int_0^{\infty} [u(\bar{c}) - u'(\bar{c})\bar{c}] e^{-\rho t} dt + u'(\bar{c}) \int_0^{\infty} C e^{-\rho t} dt < +\infty.$$

Let us define $W = \sup_{C\in\mathcal{K}} \int_0^\infty u(C)e^{-\rho t}dt$. Assume that $W > -\infty$ (otherwise the proof is trivial). Let $C_i \in \mathcal{K}$ be the maximizing sequence of $\int_0^\infty u(C)e^{-\rho t}dt$ so $\lim_{i\to\infty}\int_0^\infty u(C_i)e^{-\rho t}dt = W$.

Since $\mathcal{K}$ is relatively weak compact, suppose that $C_i \rightharpoonup C^*$ for some $C^*$ in $L^1(e^{-\rho t})$. By Mazur's Lemma, there is a sequence of convex combination

$$x_n = \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)}C_{i(n)} \to C^*, \omega_{i(n)} \geq 0, \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} = 1.$$

Because $u$ is concave, we have

$$\begin{aligned}
\limsup_{n\to\infty} u(x_n) &= \limsup_{n\to\infty} u(\sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)}C_{i(n)}) \\
&\leq \limsup_{n\to\infty} [u(C^*) + u'(C^*)(\sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)}C_{i(n)} - C^*)] = u(C^*).
\end{aligned}$$

Since this holds for almost $t$, integrate w.r.t $e^{-\rho t}dt$ to get

$$\int_0^\infty \limsup_{n\to\infty} u(x_n)e^{-\rho t}dt \leq \int_0^\infty u(C^*)e^{-\rho t}dt < +\infty.$$

Using a reverse Fatou's lemma (see Appendix) we yield

$$\limsup_{n\to\infty}\int_0^\infty u(x_n)e^{-\rho t}dt \leq \int_0^\infty \limsup_{n\to\infty} u(x_n)e^{-\rho t}dt \leq \int_0^\infty u(C^*)e^{-\rho t}dt. \tag{7}$$

Moreover, by Jensen's inequality we get

$$\limsup_{n\to\infty}\int_0^\infty u(x_n)e^{-\rho t}dt \geq \limsup_{n\to\infty}\sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)}\int_0^\infty u(C_{i(n)})e^{-\rho t}dt. \tag{8}$$

But since $\int_0^\infty u(C_{i(n)})e^{-\rho t}dt \to W$, (7) and (8) imply $\int_0^\infty u(C^*)e^{-\rho t}dt \geq W$.

So it remains to show that $C^*$ is feasible.

The task is now to show that there exists some $(K^*, L_A^*, L_Y^*, A^*, R^*, Q^*, S_R^*, S_Q^*)$ in $\mathcal{K}$ such that $(C^*, K^*, L_A^*, L_Y^*, A^*, R^*, Q^*, S_R^*, S_Q^*)$ satisfies (1)-(4).

Consider a feasible sequence $(K, L_A, L_Y, A, R, Q, S_R, S_Q)_{i(n)}$ in $\mathcal{K}$ associated with $C_{i(n)}$. According to Lemma2 and Jensen's inequality we have

$$\begin{aligned}
C^* &= \lim_{n\to\infty} x_n = \lim_{n\to\infty}\sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)}C_{i(n)} \\
&\leq \lim_{n\to\infty}\sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)}[A_i^\theta f(K_i, L_{Y_i}, Q_i, R_i) - \delta K_i - \dot{K}_i] \\
&= \lim_{n\to\infty}\sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)}[\lim_{n\to\infty} A_i^\theta f(\lim_{n\to\infty} K_{i(n)}, \lim_{n\to\infty} L_{Y_i}, Q_i, R_i) - \delta \lim_{n\to\infty} K_{i(n)}] - \lim_{n\to\infty}\sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)}\dot{K}_i \\
&\leq A^{*\theta} f(K^*, L_Y^*, \lim_{n\to\infty}\sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)}Q_i, \lim_{n\to\infty}\sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)}R_i) - \delta K^* - \lim_{n\to\infty}\sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)}\dot{K}_i \\
&= A^{*\theta} f(K^*, L_Y^*, Q^*, R^*) - \delta K^* - \dot{K}^*.
\end{aligned}$$

Therefore,

$$C^* \leq A^{*\theta} f(K^*, L_Y^*, Q^*, R^*) - \delta K^* - \dot{K}^*.$$

Applying a similar argument and using Jensen's inequality we get

$$\begin{aligned}
\dot{A}^* &= \lim_{n\to\infty}\sum_{i=n}^{\mathcal{N}(n)}\omega_{i(n)}\dot{A}_{i(n)} = \sum_{i=n}^{\mathcal{N}(n)}\omega_{i(n)}\Psi(\lim_{n\to\infty}A_i, \lim_{n\to\infty}L_{A_i}) = \Psi(A^*, L_A^*),\\
\dot{S}_R^* &= \lim_{n\to\infty}\sum_{i=n}^{\mathcal{N}(n)}\omega_{i(n)}\dot{S}_{R_i} = \lim_{n\to\infty}\sum_{i=n}^{\mathcal{N}(n)}\omega_{i(n)}(h(S_{R_i})-R) = h(S_R^*)-R^*,\\
\dot{S}_Q^* &= \lim_{n\to\infty}\sum_{i=n}^{\mathcal{N}(n)}\omega_{i(n)}\dot{S}_{Q_i} = -\lim_{n\to\infty}\sum_{i=n}^{\mathcal{N}(n)}\omega_{i(n)}Q_i = -Q^*
\end{aligned}$$

Therefore, $(C^*, K^*, L_A^*, L_Y^*, A^*, R^*, Q^*, S_R^*, S_Q^*)$ satisfies (1)-(4).

The proof is done. ■

Our proof is adapted from works of Chichilnisky (1981) and d'Albis et al (2008) to the endogenous technical change model with less stringent assumptions. The technology is not convex in our model ( d'Albis et al (2002) assumed the technology is convex w.r.t consumption). We prove that the control variable (as consumption $C$ ) and derivative of state variables weakly converge in the weak topology $\sigma(L^1(e^{-\rho t}), L^\infty)$, while the state variables pointwise converge. And for pointwise converge sequence, the continuity is all that is necessary to prove the feasibility. Therefore, concavity is not needed for the state variables. Related results on the existence of solution can be found in Chichilnisky (1981) who used the theory of Sobolev weighted space and imposed a Caratheodory condition on utility function.

Once the existence of solution is proven, the solution is unique if utility function is strictly concave and technology is convex. This enable us to derive sufficient conditions for opimality. In other words, the Kuhn-Tucker first-order conditions together with transversality conditions are necessary and sufficient conditions for an optimal solution.

Define $\sigma_C = -\frac{CU_{Cc}}{U_C}$ be the elasticity of marginal utility and $F = A^\theta f(K, L_Y, Q, R)$.

We have proven that (P) has a solution. Then the necessary conditions are characterized by the Kuhn-Tucker conditions. By setting the current-value Hamiltonian,

$$H(C, K, Q, R, L_Y, A) = u(C) + \lambda[h(S_R) - R] - \mu Q + \nu(F - C - \delta K) + \omega\Psi(A, L_A)$$

where $\lambda, \mu,\ \nu, \omega$ are four costate variables, the first order conditions ($\frac{\partial H}{\partial C} = 0,\ \frac{\partial H}{\partial Q} = 0, \frac{\partial H}{\partial R} = 0, \frac{\partial H}{\partial L_Y} = 0$) yield

$$\begin{aligned}
\nu &= U_C,\\
\mu &= vF_Q,\\
\lambda &= vF_R,\\
\omega &= \frac{vF_{L_Y}}{\Psi_{L_A}}.
\end{aligned}$$

From Euler equations $\frac{\partial H}{\partial K} = \rho\nu - \dot{\nu}$, $\frac{\partial H}{\partial S_R} = \rho\lambda - \dot{\lambda}$, $\frac{\partial H}{\partial S_Q} = \rho\mu - \dot{\mu}$, and $\frac{\partial H}{\partial A} = \rho\omega - \dot{\omega}$ we get

$$\begin{aligned}
\dot{\nu} &= (\rho - F_K + \delta)v\\
\dot{\mu} &= \rho\mu\\
\dot{\lambda} &= (\rho - h_{S_R})\lambda\\
\dot{\omega} &= (\rho - \Psi_A)\omega - vF_A.
\end{aligned}$$

The transversality conditions are

$$\lim_{t\to+\infty}\lambda S_R e^{-\rho t} = \lim_{t\to+\infty}\mu S_Q e^{-\rho t} = \lim_{t\to+\infty}\nu K e^{-\rho t} = \lim_{t\to+\infty}\omega A e^{-\rho t} = 0.$$

Moreover, it is easy to see that, at the optimum, the Ramsey conditions and Hotelling rules are satisfied.

$$\rho + \sigma_C\frac{\dot{C}^*}{C^*} = F_K - \delta = \frac{\dot{F}_Q}{F_Q} = \frac{\dot{F}_R}{F_R} + h_{S_R} = \frac{\dot{F}_{L_Y}}{F_{L_Y}} - \frac{\dot{g}_{L_A}}{g_{L_A}} + g_A + \frac{F_A g_{L_A}}{F_{L_Y}}.$$

We will get back to these conditions in the next section.

# 4 Characterization of balanced optimal growth paths

Before analyzing the full dynamic system, we look at the characterization of a balanced optimal growth path. The model in Section 2 corresponds to a system with four state variables. As Kolstad and Krautkraemer (1993) remarked, '...*it is difficult or impossible to characterize the qualitative features of a dynamic model evolving three state variables without restrictive assumptions about the functional forms of important relationships...*', we specify a set of restrictions imposed on preferences and production technology in order to have analytical results that may be tested with real data. We make the following assumptions:

H7.
$$u(C) = \begin{cases} \frac{C^{1-\varepsilon}-1}{1-\varepsilon}, \text{ if } \varepsilon \neq 1, \\ \ln C \text{ if } \varepsilon = 1. \end{cases},$$

H8. $f(L_Y, K, Q, R) = L_Y^{\gamma} K^{\xi} Q^{\alpha} R^{\beta}$ where $\gamma, \xi, \alpha, \beta \geq 0, \gamma + \xi + \alpha + \beta = 1$.

H9. $\Psi(A, L_A) = bA^{\phi} L_A$ where $b > 0, 0 < \phi \leq 1$

H10. $h(S_R) = mS_R, m > 0$.

This specification satisfies H1-H6. H9 is widely used in the literature (see, e.g., Aghion and Howitt, 1998, Jones, 2006). Assumption H10 allows for constant infinite growth for renewable which is not realistic for ecological restriction. However, it may be reasonable for a type of of non-fossil energy such as solar energy, wind energy or nuclear energy in which the stock of alternative energy can be considered as infinity.

Let $g_{\chi} = \dot{\chi}/\chi$ denote the growth rate of any variable $\chi$. We shall summarize the macroeconomic equilibrium in terms of five variables: $x = F/K, y = C/K, z = Q/S_Q, u = R/S_R, q = L_Y A^{\phi-1}, r = A^{\phi-1}$ from which other equilibrium rates $g_F, g_K, g_C, g_{L_Y}, g_{L_A}, g_A, g_Q, g_{S_Q}, g_R$, and $g_{S_R}$ can be derived as in the following proposition.

**Proposition 1** *The optimal growth rates take the following values*

$$\begin{aligned}
g_A &= b(r-q), \\
g_K &= x - y - \delta, \\
g_C &= \frac{\xi x - \delta - \rho}{\varepsilon}, \\
g_{S_Q} &= -z, \\
g_{S_R} &= m - u, \\
g_Q &= -y + \frac{b\theta r}{\xi} + \frac{m\beta + (1-\xi)\delta}{\xi}, \\
g_R &= -y + \frac{b\theta r}{\xi} + \frac{m(\beta+\xi) + (1-\xi)\delta}{\xi}, \\
g_{L_Y} &= -y + \frac{b\theta r}{\xi} + \frac{b\theta}{\gamma} q + \frac{m\beta + (1-\xi)\delta}{\xi}, \\
g_F &= \xi x - y + \frac{b\theta r}{\xi} + \frac{m\beta + \delta(\alpha+\beta+\gamma)}{\xi} - \delta, \\
g_{L_A} &= \frac{q}{q-r} g_{L_Y}.
\end{aligned}$$

**Proof.** See Appendix ■

A steady state satisfies that all rates of growth are constant. Let $\chi^*$ and $g^*_{\chi}$ denote respectively the value and the growth rate of any variable $\chi$ at the steady state.

**Proposition 2** *At the steady state, the growth rates take the following values*

$$
\begin{aligned}
g_Q^* &= g_{S_Q}^* = -y^* + \frac{b\theta r^*}{\xi} + \frac{m\beta + (1-\xi)\delta}{\xi}, \\
g_R^* &= g_{S_R}^* = -y^* + \frac{b\theta r^*}{\xi} + \frac{m(\beta+\xi) + (1-\xi)\delta}{\xi}, \\
g_F^* &= g_K^* = g_C^* = \frac{\xi x^* - \delta - \rho}{\varepsilon}, \\
g_{L_Y}^* &= g_{L_A}^* = 0, \\
g_A^* &= b(r^* - q^*),
\end{aligned}
$$

*where if* $\phi = 1$ *then*

$$
\begin{aligned}
x^* &= \frac{b\theta + m\beta + \delta(1-\xi)}{\xi(1-\xi)}, \\
y^* &= \frac{(\varepsilon - \xi)(b\theta + m\beta + \delta(1-\xi)) + \xi(1-\varepsilon)\delta + \rho]}{\varepsilon\xi(1-\xi)}, \\
q^* &= [y^* - \frac{m\beta + (1-\xi)\delta + b\theta}{\xi}]\frac{\gamma}{\theta b}, \\
r^* &= 1,
\end{aligned}
$$

*and if* $\phi < 1$ *then* $x^*, y^*, q^*, r^*$ *are given by*

$$
\begin{aligned}
\frac{\xi x^* - \delta - \rho}{\varepsilon} &= x^* - y^* - \delta, \\
(\xi - 1)x^* - y^* + \delta + \frac{b\theta q^* + m\beta + \delta(1-2\xi)}{\xi} &= 0, \qquad (9) \\
-y^* + \frac{m\beta + (1-\xi)\delta + b\theta q^*}{\xi} + \frac{b\theta}{\gamma}q^* &= 0, \qquad (10) \\
r^* &= q^*.
\end{aligned}
$$

**Proof.** See Appendix ■

**Remark 2** *We have* $z^* = -g_{S_Q}^*$, $u^* = m - g_{S_R}^*$. *It follows from transversality conditions at the steady state and the Euler equation that* $\lim_{t\to+\infty} \mu S_Q^* e^{-\rho t} = 0$ *where* $\mu = \mu(0)e^{-\rho t}$ *and* $S_Q^*(t) = S_Q^*(0)e^{g_Q^* t}$. *We then obtain* $\lim_{t\to+\infty} \mu(0)S_Q^*(0)e^{g_Q^* t} = 0$. *This implies* $g_Q^* < 0$. *Similarly, since* $\lim_{t\to+\infty} \lambda S_R e^{-\rho t} = 0$ *where* $\lambda = \lambda(0)e^{(\rho - m)t}$, *we get* $\lim_{t\to+\infty} \lambda(0)S_R^*(0)e^{(g_R^* - m)t} = 0$ *or* $g_R^* - m < 0$.

# 5 Econometric estimation

## 5.1 Estimated equations

We first study the dynamic behavior of the nonlinear system which is characterized by the behavior of the linearized system around the steady state.We shall summarize the macroeconomic equilibrium in terms of four stationary variables, $x = F/K, y = C/K, z = Q/S_Q, u = R/S_R, q = A^{\phi-1}L_Y, r = A^{\phi-1}$, from which other equilibrium rates can be derived as in Proposition 1. Let us denote $\mathbf{h} = (x, y, z, u, q, r)$. From the theory of linear approximation we know that in the neighborhood of the steady state, the dynamic behavior of the nonlinear system is characterized by the behavior of the linearized system around the steady state $\dot{\mathbf{h}} = \mathbf{J}(\mathbf{h} - \mathbf{h}^*)$ where $\mathbf{h}^* = (x^*, y^*, z^*, u^*, q^*, r^*)$ and $\mathbf{J}$ is the Jacobian matrix evaluated at the steady state, i.e.

$$
\mathbf{J} = \begin{pmatrix}
\partial\dot{x}/\partial x & \partial\dot{x}/\partial y & \partial\dot{x}/\partial z & \partial\dot{x}/\partial u & \partial\dot{x}/\partial q & \partial\dot{x}/\partial r \\
\partial\dot{y}/\partial x & \partial\dot{y}/\partial y & \partial\dot{y}/\partial z & \partial\dot{y}/\partial u & \partial\dot{y}/\partial q & \partial\dot{y}/\partial r \\
\partial\dot{z}/\partial x & \partial\dot{z}/\partial y & \partial\dot{z}/\partial z & \partial\dot{z}/\partial u & \partial\dot{z}/\partial q & \partial\dot{z}/\partial r \\
\partial\dot{u}/\partial x & \partial\dot{u}/\partial y & \partial\dot{u}/\partial z & \partial\dot{u}/\partial u & \partial\dot{u}/\partial q & \partial\dot{u}/\partial r \\
\partial\dot{q}/\partial x & \partial\dot{q}/\partial y & \partial\dot{q}/\partial z & \partial\dot{q}/\partial u & \partial\dot{q}/\partial q & \partial\dot{q}/\partial r \\
\partial\dot{r}/\partial x & \partial\dot{r}/\partial y & \partial\dot{r}/\partial z & \partial\dot{r}/\partial u & \partial\dot{r}/\partial q & \partial\dot{r}/\partial r
\end{pmatrix}.
$$

**Proposition 3** *If $\phi = 1$ then the Jacobian matrix has only one negative eigenvalue. If $\phi < 1$ then the Jacobian matrix has at most two negative eigenvalues.*

Proof. see Appendix.

**Remark 3** *Let $\mathbf{v}_i$, $i = 1, ..., 6$, denote the eigenvectors corresponding to six eigenvalues $\lambda_i$. We may write the general solution of $\dot{\mathbf{h}} = \mathbf{J}(\mathbf{h} - \mathbf{h}^*)$, where $\mathbf{h}^* = (x^*, y^*, z^*, u^*, q^*, r^*)$, as follows:*

$$\mathbf{h}(t) - \mathbf{h}^* = \sum_{i=1}^{6} a_i \mathbf{v}_i e^{\lambda_i t}$$

*where parameters $a_i$ are determined by the initial conditions $\mathbf{h}(0) - \mathbf{h}^* = \sum_{i=1}^{6} a_i \mathbf{v}_i$. The optimal path in the neighborhood of the steady state is located in the stable subspace which corresponds to the negative eigenvalues. Thus, if $\phi = 1$, we can write $\mathbf{h}(t) - \mathbf{h}^* = c_1 \mathbf{v}_1 e^{\lambda_1 t}$ where $\lambda_1 < 0$.*

We are now in a position to discuss about the empirical implication of the theoretical model. We focus our analysis on the growth rates $g_{F/K}$, $g_Q$, and $g_R$ at the transition path. We only consider the case $\phi = 1$, where all of these growth rates only depend on $x \equiv F/K$ and $y \equiv C/K$.[2] From the previous analysis for $\phi = 1$, as the components of $\mathbf{h}$ are independent of each other, we can write the approximation expressions for $x$ and $y$ as follows

$$\begin{aligned} x_t - x^* &= c_1 v_{1x} e^{\lambda_1 t}, & (11)\\ y_t - y^* &= c_1 v_{1y} e^{\lambda_1 t}, & (12)\end{aligned}$$

where $v_{1x}$ and $v_{1y}$, two components of $\mathbf{v}_1$. By using the initial conditions, we get

$$\begin{aligned} c_1 v_{1x} &= x_0 - x^* \\ c_1 v_{1y} &= y_0 - y^* \end{aligned}$$

Therefore, we obtain the following solution

$$\begin{aligned} x_t &= (1 - e^{\lambda_1 t})x^* + e^{\lambda_1 t} x_0, & (13)\\ y_t &= (1 - e^{\lambda_1 t})y^* + e^{\lambda_1 t} y_0. & (14)\end{aligned}$$

Recall that we have

$$\begin{aligned} g_x &= (\xi - 1)x + \frac{b\theta + m\beta + \delta(1-\xi)}{\xi}, \\ g_Q &= -y + \frac{b\theta}{\xi} + \frac{m\beta + (1-\xi)\delta}{\xi}, \\ g_R &= -y + \frac{b\theta}{\xi} + \frac{m(\beta+\xi) + (1-\xi)\delta}{\xi}. \end{aligned}$$

We substitute (13) and (14) in these expressions and use definitions $x_0 = F_0/K_0$, $y_0 = C_0/K_0$, and $g_x = g_{F/K} = \frac{1}{T}\ln(F_T/F_0) - \frac{1}{T}\ln(K_T/K_0)$, $g_Q = \frac{1}{T}\ln(Q_T/Q_0)$ and $g_R = \frac{1}{T}\ln(R_T/R_0)$, which are respectively the average growth rates of $F/K$, $Q$, and $R$, between 0 and $T$. This results in the transitional dynamics of $g_\chi$, $\chi = F/K, Q, R$ towards the steady-state of the economy.[3]

First, $g_{F/K}$ is given by[4]

$$\begin{aligned} g_{F/K} &= (\xi - 1)x_T + \frac{b\theta + m\beta + \delta(1-\xi)}{\xi} \\ &= \frac{b\theta + m\beta + \delta(1-\xi)}{\xi} + (\xi-1)(1-e^{\lambda_1 T})x^* + (\xi - 1)e^{\lambda_1 T} x_0 \\ &= \alpha_0 + \alpha_1 F_0/K_0 + \varepsilon_{F/K}, & (15)\end{aligned}$$

[2]It should be noticed that when $\phi < 1$, the model is not identified and then estimation becomes impossible unless some restrictions are imposed.

[3]The reason of using $g_{F/K}$ instead of $g_F$ is that the approximation of the latter contains in the right-hand side both $F/K$ and $C/K$ which are highly correlated. Hence, regression of $g_F$ on $F/K$ and $C/K$ will face a problem of multicolinearity.

[4]Index $i$ is dropped for the simplicity's sake.

with

$$\begin{aligned}\alpha_0 &= \frac{b\theta + m\beta + \delta(1-\xi)}{\xi} + (\xi - 1)(1 - e^{\lambda_1 T})x^*, \\ \alpha_1 &= (\xi - 1)e^{\lambda_1 T}.\end{aligned}$$

Concerning $g_Q$, we have

$$\begin{aligned}g_Q &= -(1 - e^{\lambda_1 T})y^* - e^{\lambda_1 T}y_0 + \frac{b\theta}{\xi} + \frac{m\beta + (1-\xi)\delta}{\xi} \\ &= \beta_0 + \beta_1 C_0/K_0 + \varepsilon_Q, \end{aligned} \tag{16}$$

where

$$\begin{aligned}\beta_0 &= -(1 - e^{\lambda_1 T})y^* + \frac{b\theta}{\xi} + \frac{m\beta + (1-\xi)\delta}{\xi}, \\ \beta_1 &= -e^{\lambda_1 T}.\end{aligned}$$

Similarly, $g_R$ is given by

$$\begin{aligned}g_R &= -y + \frac{b\theta}{\xi} + \frac{m(\beta + \xi) + (1-\xi)\delta}{\xi} \\ &= -(1 - e^{\lambda_1 T})y^* - e^{\lambda_1 T}y_0 + \frac{b\theta}{\xi} + \frac{m(\beta + \xi) + (1-\xi)\delta}{\xi} \\ &= \gamma_0 + \gamma_1 C_0/K_0 + \varepsilon_R, \end{aligned} \tag{17}$$

where

$$\begin{aligned}\gamma_0 &= \frac{b\theta}{\xi} + \frac{m(\beta + \xi) + (1-\xi)\delta}{\xi} - (1 - e^{\lambda_1 T})y^*, \\ \gamma_1 &= -e^{\lambda_1 T}.\end{aligned}$$

Equations (15), (16), and (17) represent three cross-sectional regressions where $\varepsilon_{F/K}$ and $\varepsilon_Q$ and $\varepsilon_R$ are the corresponding error terms.[5] These equations may be estimated by using Ordinary Least Squares as in Mankiw et al. (1992). However, as underlined by Islam (1995) and subsequent studies on income convergence, we will lose information contained by the data as we only need observations of the initial and final dates, i.e. dates 0 and $T$, and the sample size is then reduced to to the number of countries.

An alternative approach is to transform the model in a panel structure. In particular, we can rewrite equations (15), (16), and (17) as follows:

$$g_{(F/K)_{it}} = \alpha_0 + \alpha_1 F_{i,t-1}/K_{i,t-1} + \varepsilon_{(F/K)_{it}}, \tag{18}$$

$$g_{Q_{it}} = \beta_0 + \beta_1 C_{i,t-1}/K_{i,t-1} + \varepsilon_{Q_{it}}, \tag{19}$$

$$g_{R_{it}} = \gamma_0 + \gamma_1 C_{i,t-1}/K_{i,t-1} + \varepsilon_{R_{it}}, \tag{20}$$

where $i = 1, ..., N$, and $t = 1, ..., T$. As in most studies on income convergence (see Durlauf et al., 2005, for a survey), we use data corresponding to the five year interval period, i.e. data from 1977, 1982, 1987, 1992, and 1997. Hence the length between $t$ and $t-1$ is equal to 5 and $g_{\chi_{it}} = \frac{1}{5}\ln(\chi_t/\chi_{t-1})$. The purpose is to reduce the business cycle effect.

The sample size is 108 observations ($N = 27$, $T = 4$). In our panel data framework, the error terms $\varepsilon_{\chi_{it}}$, $\chi = F/K, Q, R$, include country and time effects, i.e. $\varepsilon_{\chi_{it}} = \mu_i + \lambda_t + u_{\chi_{it}}$ where $\mu_i$ and $\lambda_t$ denote country heterogeneity and time heterogeneity respectively, and $u_{\chi_{it}}$ is the idiosyncratic error. Moreover, the model predicts that coefficients $\alpha_1$, $\beta_1$, and $\gamma_1$ are negative.

Estimations of these equations can be obtained with standard panel methods (within estimation for fixed effects, Generalized Least Squares for random effects) which assume the *strict exogeneity* of regressors, i.e.

$$E[(F_s/K_s)\varepsilon_{(F/K)_t}] = E[(C_s/K_s)\varepsilon_{Q_t}] = E[(C_s/K_s)\varepsilon_{R_t}] = 0, \forall s, t.$$

[5]These error terms may correspond to omitted variables or unobserved factors that can affect the growth rates $g_{F/K}$, $g_Q$ and $g_R$. They can also represent measurement errors in these growth rates.

However, this assumption may be faulty as regressors can be correlated with some unobserved factors or with future values of the dependent variable. This is probably the case when we study macroeconomic data. For example, we may think that current consumption and capital stock may have some impacts not only on current income and energy consumption but also on their future values. This situation arises when regressors are *predetermined*, i.e.

$$E[(F_s/K_s)\varepsilon_{(F/K)_t}] = E[(C_s/K_s)\varepsilon_{Q_t}] = E[(C_s/K_s)\varepsilon_{R_t}] = 0, \forall s < t-1.$$

In this case, the model can be estimated by using Generalized Methods of Moments (see, e.g., Arellano and Bond, 1991, Baltagi, 2005, and Lee, 2002). This is also the approach adopted in our estimation strategy.

## 5.2 Data

The data covers twenty-seven OECD countries for the period 1977-1997.[6] Data on non-renewable energy consumption $Q$ and renewable energy consumption $R$ are collected from the International Energy Agency (IEA). Non-renewable energy consumption, $Q$, is measured as the sum of consumption of gas, and liquid fuels (in metric tons oil equivalent, toe). We assume that renewable energy consumption, $R$, corresponds to the sum of nuclear energy, hydroelectricity, geothermal energy, renewable fuels and waste, solar energy, wind energy, and energy from tide, wave, and ocean (also in toe).

Data on production $F$, consumption $C$, physical capital stock $K$, and population are collected from the Penn World Table 6.1 (Heston et al., 2002). We note that all figures, except population, are expressed in PPP and 1996 prices for an international comparison purpose. We use the population series and the series on real GDP per capita in 1996 prices (series RGDPL) to produce the volume of real GDP in 1996 prices. We compute the product between real GDP and investment share of real GDP on the one hand, and the product between real GDP and consumption share of real GDP on the other hand, which correspond respectively to investment and consumption in real terms. Calculation of physical capital stock is based on the investment series (i.e. investment share of real GDP) following the perpetual inventory method.[7] Data are used in per capita terms to neutralize the possible scale effect due to the difference in population size observed between countries.

*Table 1 here*

Descriptive statistics of the data are summarized in Table 1. Evolutions of the averages of ratios $F/K$ and $C/K$ are displayed in Figure **??**. These ratios have similar patterns, with two dips around 1982 and 1993, except that $F/K$ has stronger variation than $C/K$ (standard deviation of $F/K$, 0.121, is higher than that of $C/K$, 0.098). Series on consumptions of nonrenewable and renewable resources are presented in Figure **??**. The average of consumption of renewable energies $R$, is much lower than that of nonrenewable energies $Q$, which increased over the whole period of the study whereas the latter considerably decreased in the late 70s until the dip in 1983 and then increased thereafter.

*Figures* **??**-**??** *here*

For the estimations, we take data corresponding to the five year interval period (data from years 1977, 1982, 1987, 1992, and 1997) in order to eliminate business cycle effects as in most of empirical studies on economic convergence. Distributions of average annual growth rates (computed from these time intervals) of the output to capital ratio $F/K$, $g_{F/K}$, renewable energy consumption per capita, $g_R$, and nonrenewable energy consumption per capita, $g_Q$, are reported in Figures **??**, **??**, and **??**, respectively. The distribution of these growth rates sensitivity changes over time. We also observe a particularity that the dispersion of $g_Q$ and $g_R$ diminishes throughout the period of study.

[6]The data include Australia, Austria, Belgium, Canada, Denmark, Finland, France, Germany, Greece, Hungary, Iceland, Ireland, Italy, Japan, Korea, Luxembourg, Mexico, the Netherlands, New Zealand, Norway, Portugal, Spain, Sweden, Switzerland, Turkey, United Kingdom, and the United States.

[7]The perpetual inventory equation is $K_t = I_t + (1-\delta)K_{t-1}$ where $I_t$ is the investment flow. The initial capital stock is given by $K_0 = I_0/(g_I + \delta)$ where $g_I$ is the average geometric growth rate of investment from the initial date. The depreciation rate, $\delta$, is often set to 4 to 6%. In our paper, changing $\delta$ from 4% to 6% does not modify the qualitative conclusion.

### 5.3 Estimation results

Estimation results by GMM are reported in Table 2. As GMM use the first-difference transformation, the intercept and country effects (which are not separately identified) are deleted from the regressions and therefore not estimated. Furthermore, the assumption of predetermined regressors,

$$E[(F_s/K_s)\varepsilon_{(F/K)_t}] = E[(C_s/K_s)\varepsilon_{Q_t}] = E[(C_s/K_s)\varepsilon_{R_t}] = 0, \forall s < t-1,$$

allows us to use all values of $F_s/K_s$ and $C_s/K_s$, $s = 1, ..., t-2$, as possible instruments for $F_t/K_t$ and $C_t/K_t$, respectively. As a consequence, there are finally 81 observations used in the estimations.

*Table 2 here*

Empirical results based on OECD data confirm the prediction of the theoretical model. Indeed, estimation results show that coefficients $\alpha_1$, $\beta_1$, and $\gamma_1$ are negative as expected. Coefficients $\alpha_1$ and $\gamma_1$ are statistically significant at the 5% and the 10% levels respectively, while $\beta_1$ is insignificant. The Wald test confirms the existence of time effects in all regressions. The Sargan specification test for over-identifying restrictions (relative to the use of instrumental variables) is always satisfied in either regressions as these over-identifying restrictions are not rejected.

It should be noted that the GMM estimator is consistent with an AR(1) process for the regression residuals but not consistent with an AR(2). We use the Arellano-Bond (1991) tests to examine this issue. Test results reject the absence of autocorrelation of order 1 but do not reject that of order 2 for regressions with $g_{F/K}$ and $g_R$, suggesting that estimations are consistent in these cases. Concerning $g_Q$, the specification does not seem robust as Arellano-Bond test does not confirm (but only at the 10% level) the absence of an AR(2) process in the residuals.

# 6 Conclusion

The paper is an attempt to explore theoretically and empirically the interaction between growth, technological level, and consumptions of renewable and non-renewable resources. The necessary and sufficient conditions are provided in a general endogenous growth model. We also characterize the BGP together with the transitional dynamics for an analytical model which are shown helpful for empirical analysis by using panel data on OECD countries. Our estimation strategy is appealing since it accounts for country and time heterogeneities and it allows for a more flexible assumption about regressors than the usual assumption of strictly exogenous regressors in the standard framework.

As underlined previously, it would be of particular interest, in a further work, to study the identification issue of the empirical specification that can be derived from the theoretical model when $\phi < 1$ or from a more general model. Moreover, it would be promising to consider externalities from resource use in utility or production in order to improve the realism of the modeling. Competitive equilibrium and public policy would also require a particular attention.

# Appendix

The following theorem and lemmas are used in the proof of existence of an optimal solution.

Let $F$ be a family of scalar measurable functions on a finite measure space $(\Omega, \Sigma, \mu)$, $F$ is called uniformly integrable if $\{\int_E |f(t)|\, d\mu,\ f \in F\}$ converges uniformly to *zero* when $\mu(E) \to 0$.

**Dunford-Pettis Theorem**: *Denote $L^1(\mu)$ the set of functions $f$ such that $\int_0^\infty |f|\, d\mu < \infty$ and $K$ be a subset of $L^1(\mu)$. Then $K$ is relatively weak compact if and only if $K$ is uniformly integrable.*

The following is the reverse Fatou'Lemma when applying Fatou's lemma to the non-negative sequence given by $g - f_n$.

**Fatou's Lemma**: *Let $f_n$ be a sequence of extended real-valued measurable functions defined on a measure space $(\Omega, \Sigma, \mu)$. If there exists an integrable function $g$ on $\Omega$ such that $f_n \leq g$ for all $n$, then* $\limsup_{n\to\infty} \int_\Omega f_n d\mu \leq \int_\Omega \limsup_{n\to\infty} f_n d\mu$.

Mazur's lemma shows that any weakly convergent sequence in a normed linear space has a sequence of convex combinations of its members that converges strongly to the same limit. Because strong convergence is stronger than pointwise convergence, it is used in our proof for state variables converge pointwise to the limit obtained from weak convergence.

**Mazur's Lemma**: *Let $(X, ||\ ||)$ be a normed linear space and let $(u_n)_{n\in N}$ be a sequence in $X$ that converges weakly to some $u^*$ in X. Then there exists a function $\mathcal{N} : N \to N$ and a sequence of sets of real numbers $\{\omega_{i(n)} \mid i = n, ..., \mathcal{N}(n)\}$ such that $\omega_{i(n)} \geq 0$ and $\sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} = 1$ such that the sequence $(v_n)_{n\in N}$ defined by the convex combination $v_n = \sum_{i=n}^{\mathcal{N}(n)} \omega_{i(n)} u_i$ converges strongly in $X$ to $u^*$*,i.e.,$||v_n - u^*|| \to 0$ as $n \to \infty$.

Under assumptions H11-H13, the program of the social planner can be written as follows

$$\max \int_0^\infty u(C) e^{-\rho t} dt$$

subject to

$$\begin{aligned}
F &= A^\theta L_Y^\gamma K^\xi Q^\alpha R^\beta, \\
\dot{S}_Q &= -Q, \\
\dot{S}_R &= mS_R - R, \\
\dot{K} &= F - C - \delta K, \\
\dot{A} &= bA^\phi L_A, \\
1 &= L_A + L_Y, \\
& \quad L_{Y_0}, K_0, S_{Q_0}, S_{R_0} \text{ given.}
\end{aligned}$$

**Proof of Proposition 1**

**Proof.** The current-value Hamiltonian is

$$\begin{aligned}
H(C, K, Q, R, L_Y, A) &= u(C) + \lambda(mS_R - R) - \mu Q \\
&\quad + \nu(F - C - \delta K) + \omega b A^\phi (1 - L_Y)
\end{aligned}$$

where $\lambda, \mu, \nu, \omega$ are four costate variables.

The first order conditions $\frac{\partial H}{\partial C} = 0$, $\frac{\partial H}{\partial Q} = 0$, $\frac{\partial H}{\partial R} = 0$, $\frac{\partial H}{\partial L_Y} = 0$ yield

$$\nu = U_C, \tag{21}$$

$$\mu = vF_Q, \tag{22}$$

$$\lambda = vF_R, \tag{23}$$

$$\omega = \frac{vF_{L_Y}}{bA^\phi}. \tag{24}$$

From Euler equations $\frac{\partial H}{\partial K} = \rho\nu - \dot{\nu}$, $\frac{\partial H}{\partial S_R} = \rho\lambda - \dot{\lambda}$, $\frac{\partial H}{\partial S_Q} = \rho\mu - \dot{\mu}$, and $\frac{\partial H}{\partial A} = \rho\omega - \dot{\omega}$ we get

$$\begin{aligned}
\frac{\dot{\nu}}{v} &= \rho - F_K - \delta && (25)\\
\frac{\dot{\mu}}{\mu} &= \rho && (26)\\
\frac{\dot{\lambda}}{\lambda} &= \rho - m && (27)\\
\dot{\omega} &= (\rho - b\phi A^{\phi-1}(1-L_Y))\omega - vF_A.
\end{aligned}$$

By (21) and $\dot{A}/A = bA^{\phi-1}(1-L_Y)$ we get

$$\dot{\omega} = (\rho - \phi g_A)\omega - U_C F_A. \tag{28}$$

The transversality conditions are

$$\lim_{t\to+\infty} \lambda S_R e^{-\rho t} = \lim_{t\to+\infty} \mu S_Q e^{-\rho t} = \lim_{t\to+\infty} \nu K e^{-\rho t} = \lim_{t\to+\infty} \omega A e^{-\rho t} = 0.$$

From the identities $\dot{S}_R = mS_R - R$, $\dot{S}_Q = -Q$ and $\dot{K} = F - C - \delta K$, we obtain

$$\begin{aligned}
g_{S_Q} &= -z,\\
g_{S_R} &= m - u,\\
g_K &= x - y - \delta,\\
g_A &= b(r-q).
\end{aligned}$$

Since $F = A^\theta L_Y^\gamma K^\xi Q^\alpha R^\beta$, we have

$$F_K = \xi F/K = \xi x,$$

$$\begin{aligned}
\frac{\dot{F}_Q}{F_Q} &= \theta g_A + \gamma g_{L_Y} + \xi g_K + (\alpha-1)g_Q + \beta g_R, && (29)\\
\frac{\dot{F}_R}{F_R} &= \theta g_A + \gamma g_{L_Y} + \xi g_K + \alpha g_Q + (\beta-1)g_R, && (30)\\
\frac{\dot{F}_{L_Y}}{F_{L_Y}} &= \theta g_A + (\gamma-1)g_{L_Y} + \xi g_K + \alpha g_Q + \beta g_R && (31)
\end{aligned}$$

Equation (21) together with (25) yield

$$\rho - \frac{\dot{U}_C}{U_C} = F_K - \delta = \xi x - \delta. \tag{32}$$

It is easy to check that

$$\frac{\dot{U}_C}{U_C} = -\varepsilon\frac{\dot{C}}{C} = -\varepsilon g_C. \tag{33}$$

Thus,

$$g_C = \frac{\xi x - \delta - \rho}{\varepsilon}.$$

By logarithmic differentiation (22) and together with (26) we have

$$\frac{\dot{F}_Q}{F_Q} = \rho - \frac{\dot{U}_C}{U_C} = \xi x - \delta.$$

From (23) and (27) we get

$$\frac{\dot{F}_R}{F_R} = \rho - \frac{\dot{U}_C}{U_C} - m = \xi x - \delta - m.$$

From (24) and (28) we get

$$\begin{aligned}
\frac{\dot{U}_C}{U_C} + \frac{\dot{F}_{L_Y}}{F_{L_Y}} - \phi g_A &= \rho - \phi g_A - \frac{U_C F_A}{\omega} = \\
\rho - \phi g_A - \frac{U_C F_A}{U_C F_{L_Y}} A^\phi &= \rho - \phi g_A - \frac{\theta F/A}{\gamma F/L_Y} A^\phi = \\
\rho - \phi g_A - \frac{\theta}{\gamma} L_Y A^{\phi-1} &= \rho - \phi g_A - \frac{b\theta}{\gamma} q.
\end{aligned}$$

Thus,

$$\frac{\dot{F}_{L_Y}}{F_{L_Y}} = \rho - \frac{\dot{U}_C}{U_C} - \frac{b\theta}{\gamma} q = \xi x - \delta - \frac{b\theta}{\gamma} q.$$

Now, it follows from (29)-(31) that we have a system of equations with three variables $g_Q$, $g_R$, and $g_{L_Y}$:

$$\begin{aligned}
\gamma g_{L_Y} + (\alpha - 1) g_Q + \beta g_R &= \xi y - b\theta(r - q) + (\xi - 1)\delta = T_1, && (34) \\
\gamma g_{L_Y} + \alpha g_Q + (\beta - 1) g_R &= \xi y - m - b\theta(r - q) + (\xi - 1)\delta, && (35) \\
(\gamma - 1) g_{L_Y} + \alpha g_Q + \beta g_R &= \xi y - \frac{b\theta}{\gamma} q - b\theta(r - q) + (\xi - 1)\delta. && (36)
\end{aligned}$$

From (34)-(35) we have $-g_Q + g_R = m$.

From (34)-(36) we get $\gamma g_{L_Y} = b\theta q + \gamma g_Q$. Replacing this equation into (34), we have

$$(\gamma + \alpha - 1) g_Q + \beta g_R = T_1 - b\theta q$$

We then have two equations to find $g_Q$ and $g_R$:

$$\begin{aligned}
g_Q &= \frac{m\beta + b\theta q - T_1}{\xi} = \frac{b\theta r - \xi y + m\beta + (1 - \xi)\delta}{\xi}, \\
g_R &= g_Q + m = \frac{b\theta r - \xi y + m(\beta + \xi) + (1 - \xi)\delta}{\xi},
\end{aligned}$$

and

$$g_{L_Y} = \frac{b\theta}{\gamma} q + g_Q = \frac{b\theta r - \xi y + m\beta + (1 - \xi)\delta}{\xi} + \frac{b\theta}{\gamma} q.$$

By log-differentiating $F = A^\theta L_Y^\gamma K^\xi Q^\alpha R^\beta$ we get

$$\begin{aligned}
g_F &= \theta g_A + \gamma g_{L_Y} + \xi g_K + \alpha g_Q + \beta g_R = \\
&= \gamma g_{L_Y} + (\alpha - 1) g_Q + \beta g_R + \theta g_A + \xi g_K + g_Q \\
&= T_1 + \theta g_A + \xi g_K + g_Q \\
&= \xi y + (\xi - 1)\delta + \xi(x - y - \delta) + \frac{b\theta r - \xi y + m\beta + (1 - \xi)\delta}{\xi} \\
&= \xi x - y + \frac{b\theta r}{\xi} + \frac{m\beta + \delta(1 - 2\xi)}{\xi} \\
&= \xi x - y + \frac{b\theta r}{\xi} + \frac{m\beta + \delta(\alpha + \beta + \gamma)}{\xi} - \delta.
\end{aligned}$$

Finally, since $L_Y = \frac{q}{r}$,

$$g_{L_A} = \frac{\dot{L}_A}{L_A} = -\frac{\dot{L}_Y}{1 - L_Y} = \frac{L_Y}{L_Y - 1} g_{L_Y} = \frac{q}{q - r} g_{L_Y}.$$

■

**Proof of Proposition 2**

**Proof.** At the steady state, $g_A^* = b(r^* - q^*)$ is constant. Therefore, since $g_C^*$ and $g_K^*$ are constant, it follows that $x^*$ and $y^*$ are constant. It also follows from constant $g_Q^*$ that $r^*$ is constant. Thus, $q^*$, $z^*$, and $u^*$ are also constant. Since $x = F/K$ and $y = C/K$, we have $g^*_C = g^*_F = g^*_K$. Moreover, $L^*_Y = q^*/r^*$ is constant, which implies that $L^*_A$ is constant. So, we get

$$g^*_{L_Y} = g^*_{L_A} = 0.$$

Since $r^* = A^{*\phi-1}$ is constant, we have $(\phi - 1)g^*_A = 0$ or

$$(\phi - 1)b(r^* - q^*) = 0.$$

This equation together with $g^*_C = g^*_K$, $g^*_F = g^*_K$, and $g^*_{L_Y} = 0$, yield

$$\begin{aligned}
\frac{\xi x^* - \delta - \rho}{\varepsilon} &= x^* - y^* - \delta, \\
\xi x^* - y^* + \frac{b\theta r^* + m\beta + \delta(1 - 2\xi)}{\xi} &= x^* - y^* - \delta, \\
-y^* + \frac{m\beta + (1 - \xi)\delta + b\theta r^*}{\xi} + \frac{b\theta}{\gamma} q^* &= 0, \\
(\phi - 1)(r^* - q^*) &= 0.
\end{aligned}$$

We consider two cases:

a) If $\phi = 1$, $r^* = A^{*\phi-1} = 1$, we have

$$\begin{aligned}
\frac{\xi x^* - \delta - \rho}{\varepsilon} &= x^* - y^* - \delta \\
\xi x^* - y^* + \frac{b\theta + m\beta + \delta(1 - 2\xi)}{\xi} &= x^* - y^* - \delta \\
-y^* + \frac{m\beta + (1 - \xi)\delta + b\theta}{\xi} + \frac{b\theta}{\gamma} q^* &= 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
x^* &= \frac{b\theta + m\beta + \delta(1 - \xi)}{\xi(1 - \xi)}, \\
y^* &= \frac{(\varepsilon - \xi)(b\theta + m\beta + \delta(1 - \xi)) + \xi(1 - \varepsilon)\delta + \rho]}{\varepsilon\xi(1 - \xi)}, \\
q^* &= [y^* - \frac{m\beta + (1 - \xi)\delta + b\theta}{\xi}]\frac{\gamma}{\theta b}.
\end{aligned}$$

b) If $\phi \neq 1$, $A^* = (r^*)^{1/\phi-1}$, and then $r^* = q^*$. Hence, we have three equations which determine the optimal growth rates at the steady state

$$\begin{aligned}
\frac{\xi x^* - \delta - \rho}{\varepsilon} &= x^* - y^* - \delta, \\
\xi x^* + \frac{b\theta q^* + m\beta + \delta(1 - 2\xi)}{\xi} &= x^* - \delta, \\
-y^* + \frac{m\beta + (1 - \xi)\delta + b\theta q^*}{\xi} + \frac{b\theta}{\gamma} q^* &= 0.
\end{aligned}$$

Finally, since $\dot{S}_Q/S_Q = -Q/S_Q$ and $g^*_{S_Q}$ is constant, we have $g^*_Q = g^*_{S_Q}$. Similarly, we have $g^*_R = g^*_{S_R}$.
■

**Proof of Proposition 3**

**Proof.** *Case 1.* If $\phi = 1$. In this case $r = 1$, we just need to analyze the dynamic system of $x, y, z, u, q$.

By logarithmic differentiation and using Proposition 1 we get

$$
\begin{aligned}
\dot{x} &= (g_F - g_K)x = [(\xi-1)x + \frac{b\theta}{\xi} + \frac{m\beta + \delta(1-\xi)}{\xi}]x, \\
\dot{y} &= (g_C - g_K)y = [\frac{(\xi-\epsilon)x + (\varepsilon-1)\delta - \rho}{\varepsilon} + y]y, \\
\dot{z} &= (g_Q + z)z = (-y + \frac{b\theta}{\xi} + \frac{m\beta + (1-\xi)\delta}{\xi} + z)z, \\
\dot{u} &= (g_R + u - m)u = (-y + \frac{b\theta}{\xi} + \frac{m\beta + (1-\xi)\delta}{\xi} + u)u, \\
\dot{q} &= (g_{L_Y})q = (-y + \frac{b\theta}{\xi} + \frac{b\theta}{\gamma}q + \frac{m\beta + (1-\xi)\delta}{\xi})q.
\end{aligned}
$$

The dynamics of $\mathbf{h} = (x, y, z, u, q, r)$ is described by the system above. From the theory of linear approximation we know that in the neighborhood of the steady state, the dynamic behavior of the nonlinear system is characterized by the behavior of the linearized system around the steady state $\dot{\mathbf{h}} = \mathbf{J}(\mathbf{h} - \mathbf{h}^*)$ where $\mathbf{h}^* = (x^*, y^*, z^*, u^*, q^*, r^*)$ and $\mathbf{J}$ is the Jacobian matrix evaluated at the steady state, i.e.

$$
\mathbf{J} = \begin{pmatrix}
\partial\dot{x}/\partial x & \partial\dot{x}/\partial y & \partial\dot{x}/\partial z & \partial\dot{x}/\partial u & \partial\dot{x}/\partial q \\
\partial\dot{y}/\partial x & \partial\dot{y}/\partial y & \partial\dot{y}/\partial z & \partial\dot{y}/\partial u & \partial\dot{y}/\partial q \\
\partial\dot{z}/\partial x & \partial\dot{z}/\partial y & \partial\dot{z}/\partial z & \partial\dot{z}/\partial u & \partial\dot{z}/\partial q \\
\partial\dot{u}/\partial x & \partial\dot{u}/\partial y & \partial\dot{u}/\partial z & \partial\dot{u}/\partial u & \partial\dot{u}/\partial q \\
\partial\dot{q}/\partial x & \partial\dot{q}/\partial y & \partial\dot{q}/\partial z & \partial\dot{q}/\partial u & \partial\dot{q}/\partial q
\end{pmatrix}.
$$

Note that $x^*, y^*, z^*, u^*, q^*$ are stationary variables, i.e. if $\dot{x} = f(\mathbf{h})x$ then $f(\mathbf{h}^*) = 0$ and

$$
\frac{\partial\dot{x}}{\partial x}(\mathbf{h}^*) = \frac{\partial f(\mathbf{h}^*)}{\partial x}x^*.
$$

Thus, we get the Jacobian matrix

$$
\mathbf{J} = \begin{pmatrix}
(\xi-1)x^* & 0 & 0 & 0 & 0 \\
(\frac{\xi-\varepsilon}{\varepsilon})y^* & y^* & 0 & 0 & 0 \\
0 & -z^* & z^* & 0 & 0 \\
0 & -u^* & 0 & u^* & 0 \\
0 & -q^* & 0 & 0 & \frac{b\theta}{\gamma}q^*
\end{pmatrix}.
$$

The characteristic roots $\lambda_k$, $k = 1, .., 5$, are the solutions of the characteristic equation $|\mathbf{J} - \lambda\mathbf{U}| = 0$ where $\mathbf{U}$ is the $5 \times 5$ unit matrix. We can write at $\mathbf{h}^*$ that

$$
[\frac{b\theta}{\gamma}q^* - \lambda_5][u^* - \lambda_4][z^* - \lambda_3][y^* - \lambda_2][((\xi-1)x^* - \lambda_1] = 0
$$

It is easy to see that there is only $\lambda_1 = (\xi-1)x^* < 0$ while the others are positive.

*Case 2.* If $\phi \neq 1$, we must analyze the dynamic system of $x, y, z, u, q$ and $r = A^{\phi-1}$. Since $\dot{r}/r = (\phi-1)g_A$, we know that $g_A^* = 0$ which implies that $r$ is a stationary variable. Moreover, $r^* = q^*$. We have

$$
\begin{aligned}
\dot{x} &= (g_F - g_K)x = [(\xi-1)x + \frac{b\theta r}{\xi} + \frac{m\beta + \delta(1-\xi)}{\xi}]x, \\
\dot{y} &= (g_C - g_K)y = [\frac{(\xi-\epsilon)x + (\varepsilon-1)\delta - \rho}{\varepsilon} + y]y, \\
\dot{z} &= (g_Q + z)z = (-y + \frac{b\theta r}{\xi} + \frac{m\beta + (1-\xi)\delta}{\xi} + z)z, \\
\dot{u} &= (g_R + u - m)u = (-y + \frac{b\theta r}{\xi} + \frac{m\beta + (1-\xi)\delta}{\xi} + u)u, \\
\dot{q} &= ((\phi-1)g_A + g_{L_Y})q = ((\phi-1)b(r-q) - y + \frac{b\theta r}{\xi} + \frac{b\theta}{\gamma}q + \frac{m\beta + (1-\xi)\delta}{\xi})q, \\
\dot{r} &= [(\phi-1)g_A]r = [b(\phi-1)(r-q)]r.
\end{aligned}
$$

It is easy to get

$$J = \begin{pmatrix} (\xi-1)x^* & 0 & 0 & 0 & 0 & \frac{b\theta}{\xi}x^* \\ \frac{(\xi-\epsilon)}{\varepsilon}y^* & y^* & 0 & 0 & 0 & 0 \\ 0 & -z^* & z^* & 0 & 0 & \frac{b\theta}{\xi}z^* \\ 0 & -u^* & 0 & u^* & 0 & \frac{b\theta}{\xi}u^* \\ 0 & -q^* & 0 & 0 & [(1-\phi)b+\frac{b\theta}{\gamma}]q^* & [(\phi-1)b+\frac{b\theta}{\xi}]q^* \\ 0 & 0 & 0 & 0 & b(1-\phi)r^* & b(\phi-1)r^* \end{pmatrix}.$$

The characteristic roots $\lambda_k$, $k=1,..,6$ are the solutions of the characteristic equation

$$|\mathbf{J}-\lambda\mathbf{U}|=0 \tag{37}$$

where $\mathbf{U}$ is the $6\times 6$ unit matrix. Equation (37) is equivalent to

$$(z^*-\lambda)(u^*-\lambda)\det M=0$$

where

$$M = \begin{pmatrix} (\xi-1)x^*-\lambda & 0 & 0 & \frac{b\theta}{\xi}x^* \\ \frac{(\xi-\epsilon)}{\varepsilon}y^* & y^*-\lambda & 0 & 0 \\ 0 & -q^* & [(1-\phi)b+\frac{b\theta}{\gamma}]q^*-\lambda & [(\phi-1)b+\frac{b\theta}{\xi}]q^* \\ 0 & 0 & b(1-\phi)r^* & b(\phi-1)r^*-\lambda \end{pmatrix}.$$

We then get two positive solutions, $\lambda=z^*$ and $\lambda=u^*$ immediately. We have

$$\begin{aligned} \det M(\lambda) &= ((\xi-1)x^*-\lambda)(y^*-\lambda)\begin{vmatrix} ((1-\phi)b+\frac{b\theta}{\gamma})q^*-\lambda & ((\phi-1)b+\frac{b\theta}{\xi})q^* \\ b(1-\phi)r^* & b(\phi-1)r^*-\lambda \end{vmatrix} \\ &\quad -\frac{b\theta}{\xi}x^*\begin{vmatrix} \frac{(\xi-\epsilon)}{\varepsilon}y^* & y^*-\lambda & 0 \\ 0 & -q^* & [(1-\phi)b+\frac{b\theta}{\gamma}]q^*-\lambda \\ 0 & 0 & b(1-\phi)r^* \end{vmatrix} \\ &= ((\xi-1)x^*-\lambda)(y^*-\lambda)\det N(\lambda)+\frac{b\theta}{\xi}\frac{(\xi-\varepsilon)}{\varepsilon}b(1-\phi)x^*y^*q^*r^*, \end{aligned}$$

where

$$\det N(\lambda) = \begin{vmatrix} ((1-\phi)b+\frac{b\theta}{\gamma})q^*-\lambda & ((\phi-1)b+\frac{b\theta}{\xi})q^* \\ b(1-\phi)r^* & b(\phi-1)r^*-\lambda \end{vmatrix}.$$

Hence, $\det M(\lambda)$ is a polynomial of degree 4,

$$\det M(\lambda)=H(\lambda)=\lambda^4+c_1\lambda^3+c_2\lambda^2+c_3\lambda^2+c_4\lambda+c_5$$

Let $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ be solutions of $H(\lambda)=0$. Then by Viete's theorem, we get

$$\begin{aligned} H(0) &= \lambda_1\lambda_2\lambda_3\lambda_4= \\ &= (\xi-1)x^*y^*[(\frac{b\theta}{\gamma}+\frac{b\theta}{\xi})b(\phi-1)r^*q^*]-\frac{b\theta}{\xi}x^*\frac{(\xi-\varepsilon)}{\varepsilon}y^*b(\phi-1)q^*r^* \\ &= [b(\phi-1)r^*q^*b\theta x^*y^*][\frac{(\xi-1)(\xi+\gamma)}{\gamma}-\frac{(\xi-\varepsilon)}{\varepsilon}] \\ &= [b(\phi-1)r^*q^*b\theta x^*y^*][\frac{(\xi-1)\xi\varepsilon+(\xi-1)\gamma\varepsilon-\gamma(\xi-\varepsilon)}{\gamma\varepsilon}] \\ &= [b(\phi-1)r^*q^*b\theta x^*y^*][\frac{(\xi-1)\xi\varepsilon+\xi\gamma(\varepsilon-1)}{\gamma\varepsilon}] \\ &= [b(1-\phi)\theta r^*q^*bx^*y^*][\frac{\xi(\alpha+\beta)\varepsilon+\gamma)}{\gamma\varepsilon}]>0. \end{aligned}$$

It follows from (9) and (10) that

$$(\xi-1)x^*+y^*=\frac{b\theta q^*}{\gamma}.$$

Therefore, $\lambda_1+\lambda_2+\lambda_3+\lambda_4=(\xi-1)x^*+y^*+[(1-\phi)b+\frac{b\theta}{\gamma}]q^*+b(\phi-1)r^*=\frac{2b\theta q^*}{\gamma}>0$ and equation $H(\lambda)=0$ has either two negative solutions or zero negative solution.

If $\xi<\varepsilon$, it follows that $H((\xi-1)x^*)=\frac{b\theta}{\xi}\frac{(\xi-\varepsilon)}{\varepsilon}b(1-\phi)x^*y^*q^*r^*<0$.

It results that $H((\xi-1)x^*)H(0)<0$ and there is a negative $\lambda\in((\xi-1)x^*,0)$. Since $\lim_{\lambda\to-\infty}H(\lambda)>0$, there is another negative $\lambda\in(-\infty,(\xi-1)x^*)$. ■

## References


[1] Aghion P., Howitt P. (1998), *Endogenous Growth Theory*, MIP Press, Cambridge, MA.

[2] d'Albis H., Gourdel P., Le Van C. (2008), "Existence of solution in continuous-time optimal growth model", *Economic Theory* 37, 321–333.

[3] Alvarez F., Marrero G.A., Puch L.A. (2005), "Air pollution and the macroeconomy across European countries", FEDEA Working paper n°2005-10.

[4] André F.J., Cerdá E. (2006), "On natural resource substitution", *Resources Policy* 30, 233–246.

[5] Arellano M., Bond S. (1991), "Some tests of specification for panel data: Monte Carlo evidence and an application to employment equations", *Review of Economic Studies* 58, 227–297.

[6] Baltagi B.H. (2005), *Econometric Analysis of Panel Data*, John Wiley & Sons, Chichester, 3rd edition.

[7] Barbier, E.B. (1999): Endogenous Growth and Natural Resource Scarcity, Environmental and Resource Economics, 14, 51-74.

[8] Bretschger L. (2005), "Economics of technological change and the natural environment: How effective are innovations as a remedy for resource scarcity?", *Ecological Economics* 54, 148–163.

[9] Bretschger L. (2006), "Energy prices, growth, and the channels in between: Theory and evidence", ETH Zurich working paper n°06/47.

[10] Brock W.A., Taylor M.S. (2004), "The green Solow model", NBER Working Paper n°10557.

[11] Brock W.A., Taylor M.S. (2006), "Economic growth and the environment: A review of theory and empirics", in Aghion P., Durlauf S.N. (eds), *Handbook of Economic Growth*, volume 1B, chapter 28, North Holland, Amsterdam.

[12] Chichilnisky, G. (1981), Existence and characterization of optimal growth paths including models with non-convexities in utilities and technologies, *Review of Economic Studies*, 48, 51-61.

[13] Durlauf S.N., Johnson P.A., Temple J.R.W. (2005), "Growth econometrics", in Aghion P., Durlauf S.N. (eds), *Handbook of Economic Growth*, volume 1A, chapter 8, North Holland, Amsterdam.

[14] Gerlagh R., van der Zwaan B. (2003), "Gross world product and consumption in a global warming model with endogenous technological change", *Resource and Energy Economics* 25, 35–57.

[15] Grimaud A., Rougé L. (2004), "Polluting non-renewable vs. non-polluting renewable resources: environmental policy and growth", Toulouse School of Economics mimeo.

[16] Grimaud A., Rougé L. (2005), "Polluting non-renewable resources, innovation and growth: welfare and environmental policy", *Resource and Energy Economics* 27, 109–129.

[17] Grimaud A., Rougé L. (2008), "Environment, directed technical change and economic policy", *Environmental and Resource Economics* 41, 439–463.

[18] Groth, C., Schou P. (2002), " Can non-renewable resources alleviate the knife-edge character of endogenous growth", *Oxford Economics Papers* 54, 386–411.

[19] Groth C., Schou P. (2007), " Growth and non-renewaable resources: The different roles of capital and resource taxes", *Journal of Environmental and Economics and Management* 53, 80–98.

[20] Growiec J., Schumacher I. (2008), "On technical change in the elasticities of resource inputs", *Resources Policy* 33, 210–221.

[21] Heston A., Summers R., Aten B. (2002), "Penn world table version 6.1", Center for International Comparisons, University of Pennsylvania (CICUP).

[22] Islam N. (1995), "Growth empirics: A panel data approach", *The Quarterly Journal of Economics* 110, 1127–1170.

[23] Jones C.I. (2006), "Growth and ideas", in Aghion P., Durlauf S.N. (eds), *Handbook of Economic Growth*, volume 1B, chapter 16, North Holland, Amsterdam.

[24] Kolstad C.D., Krautkraemer J.A. (1993), "Natural resource use and the environment", in Kneese A.V., Sweeney J.L. (eds), *Handbook of Natural Resource and Energy Economics*, volume III, chapter 26, North Holland, Amsterdam.

[25] Lee M.-j. (2002), *Panel Data Econometrics: Methods of Moments and Limited Dependent Variables*, Academic Press, San Diego.

[26] Mankiw N.G., Romer D., Weil D.N. (1992), "A contribution to the empirics of economic growth", *The Quarterly Journal of Economics* 107(2), 407–437.

[27] Miketa A., Mulder P. (2005), "Energy productivity across developed and developing countries in 10 manufacturing sectors: Patterns of growth and convergence", *Energy Economics* 27, 429–453.

[28] Mulder P., De Groot H.L.F. (2007), "Sectoral energy- and labour-productivity convergence", *Environmental and Resource Economics* 36, 85–112.

[29] Scholz, C.M. and G. Ziemes (1999): Exhaustible Resources, Monopolistic Competition, and Endogenous Growth, Environmental and Resource Eco- nomics, 13, 169-185.

[30] Smulders S. (2005), "Endogenous technological change, natural resources, and growth", in Simpson R.D., Toman M.A., Ayres R.U. (eds), *Scarcity and Growth Revisited: Natural Resources and the Environment in the New Millennium*, chapter 8, Johns Hopkins University Press.

[31] Tahvonen O., Salo S. (2001), "Economic growth and transition between renewable and nonrenewable energy resources", *European Economic Review* 45, 1379–1398.

[32] Tsur Y., Zemel A. (2003), "Optimal transition to backstop substitutes for nonrenewable resources", *Journal of Economic Dynamics & Control* 27, 551–572.

Table 1: Descriptive statistics, 1977–1997

| Variable | Mean | Std. Dev. | Min | Max |
|---|---|---|---|---|
| Consumption per capita | 10.558 | 3.473 | 2.757 | 21.505 |
| Capital stock per capita | 46.629 | 19.066 | 5.247 | 89.764 |
| GDP per capita | 16.592 | 5.591 | 4.325 | 37.917 |
| Consumption-capital ratio $C/K$ | 0.254 | 0.098 | 0.129 | 0.730 |
| Output-capital ratio $F/K$ | 0.391 | 0.121 | 0.190 | 0.911 |
| Nonrenewable energy consumption $Q$ | 3.145 | 1.706 | 0.496 | 10.637 |
| Renewable energy consumption $R$ | 0.864 | 1.097 | 0.004 | 5.467 |
| Number of countries | 27 | | | |
| Number of years | 21 | | | |

Notes: Data on energy consumption are collected from the IEA for the period 1977–1997. Consumptions of renewable and nonrenewable energies are expressed in metric tons oil equivalent (toe). Economic data are drawn from the Penn World Table 6.1 (see Heston et al., 2002). GDP, consumption, and capital stock are measured in thousands U.S. dollars and 1996 prices. All figures are in per capita terms.

Table 2: Estimation results

| Equation | Variable | Coefficient | Sargan | AR(1) | AR(2) | Wald |
|---|---|---|---|---|---|---|
| $g_{(F/K)_t}$ | $(F/K)_{t-1}$ | -0.174$^{**}$ (0.060) | 0.229 | -3.08$^{**}$ | -1.15 | 17.4$^{**}$ |
| $g_{Q_t}$ | $(C/K)_{t-1}$ | -0.015 (0.067) | 0.883 | -2.86$^{**}$ | 1.41$^{*}$ | 17.8$^{**}$ |
| $g_{R_t}$ | $(C/K)_{t-1}$ | -0.275$^{*}$ (0.166) | 5.7 | -2.37$^{**}$ | -0.553 | 6.52$^{*}$ |

Notes: Regressions include country effects and year effects. Over-identifying restrictions are tested by the Sargan test. AR(1) and AR(2) tests are the Arellano and Bond (1991) tests for serial correlation of order 1 and 2 respectively. The Wald test is for significance of year dummies. Estimation results are obtained by GMM with robust standard error *à la White* given in parentheses. $^{*}$ and $^{**}$ represent significance levels of 10% and 5% respectively.

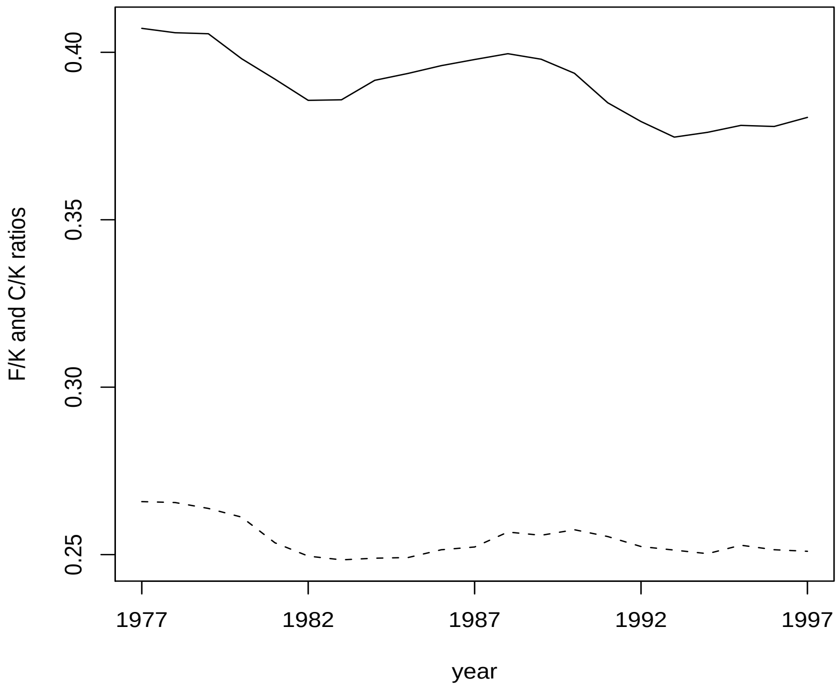


Figure 1: Averages of ratios $F/K$ (solid line) and $C/K$ (dashed line), period 1977–1997.


Figure 2: Average consumptions per capita of nonrenewable energies $Q$ (solid line) and renewable energies $R$ (dashed line), period 1977–1997.

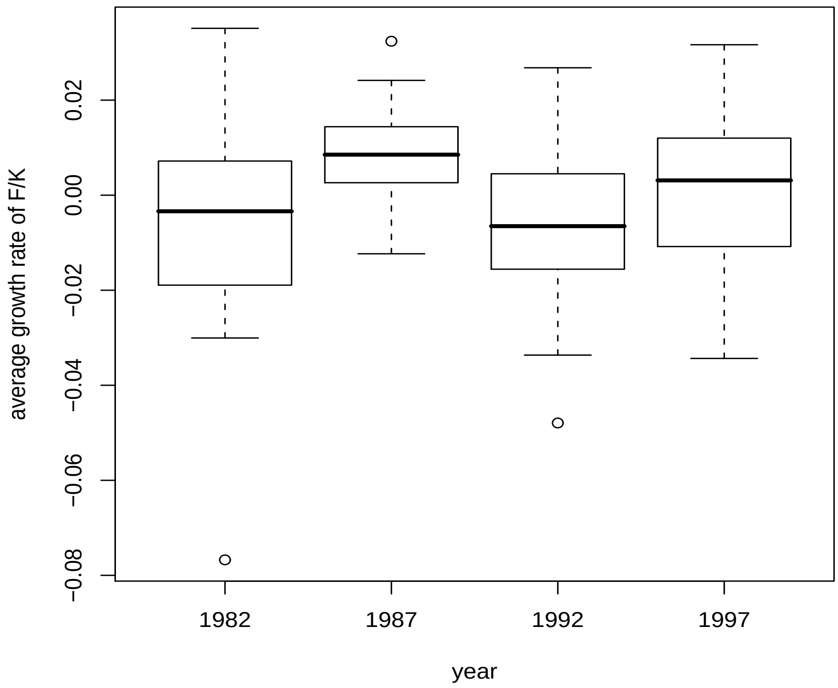


Figure 3: Average growth rate of $F/K$, period 1977–1997. The box plots relative to 1982, 1987, 1992, and 1997 represent the distribution of the average annual growth rate observed for periods 1977-1982, 1982-1987, 1987-1992, and 1992-1997, respectively.

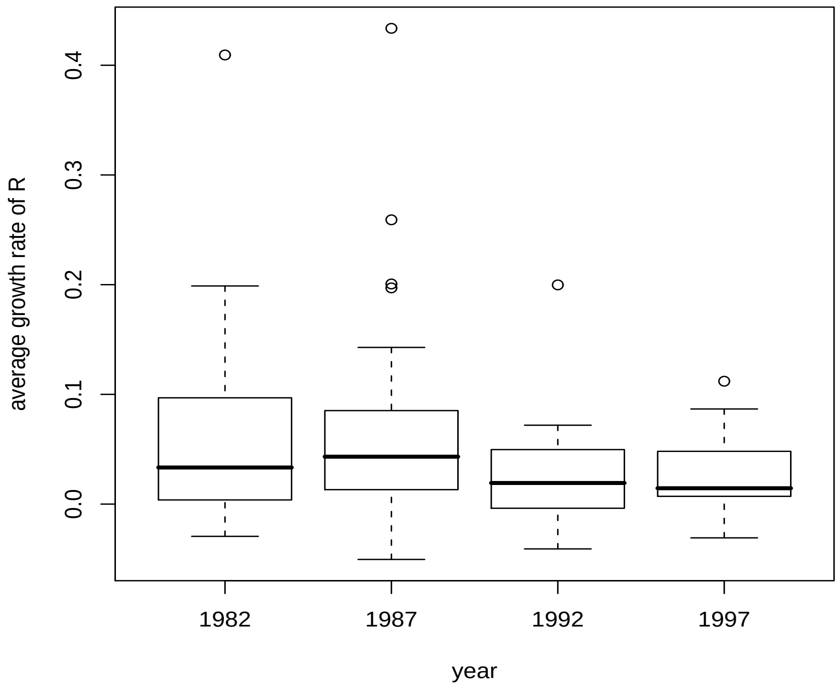


Figure 4: Average growth rate of renewable resource consumption per capita $g_R$, period 1977–1997. The box plots relative to 1982, 1987, 1992, and 1997 represent the distribution of the average annual growth rate observed for periods 1977-1982, 1982-1987, 1987-1992, and 1992-1997, respectively.

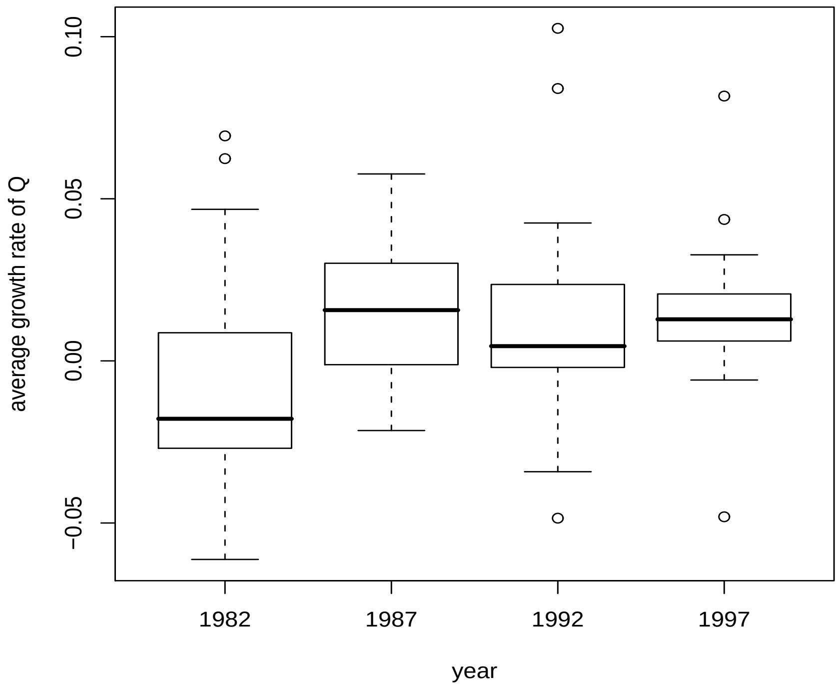


Figure 5: Average growth rate of nonrenewable resource consumption per capita $g_Q$, period 1977–1997. The box plots relative to 1982, 1987, 1992, and 1997 represent the distribution of the average annual growth rate observed for periods 1977-1982, 1982-1987, 1987-1992, and 1992-1997, respectively.