

Exhaustible Resources, Technology Transition, And Endogenous Fertility in an Overlapping – Generations Model.

Nguyen Manh Hung *
Nguyen Van Quyen **

* Département économique, Université Laval, Cité Universitaire, Ste Foy, Québec, Canada G1K 7P4 ; nmhung@ecn.ulaval.ca.

**Département de science économique, Université d'Ottawa, 55 Laurier E, Ottawa, Ontario, Canada, K1N 6N5; nvquyen@uottawa.ca.

The DEPOCEN WORKING PAPER SERIES disseminates research findings and promotes scholar exchanges in all branches of economic studies, with a special emphasis on Vietnam. The views and interpretations expressed in the paper are those of the author(s) and do not necessarily represent the views and policies of the DEPOCEN or its Management Board. The DEPOCEN does not guarantee the accuracy of findings, interpretations, and data associated with the paper, and accepts no responsibility whatsoever for any consequences of their use. The author(s) remains the copyright owner.

DEPOCEN WORKING PAPERS are available online at <http://www.depocenwp.org>

EXHAUSTIBLE RESOURCES, TECHNOLOGY TRANSITION, AND ENDOGENOUS FERTILITY IN AN OVERLAPPING-GENERATIONS MODEL

NGUYEN MANH HUNG * AND NGUYEN VAN QUYEN **

The paper presents a synthesis of the economics of exhaustible resources and that of endogenous fertility in an overlapping-generations model. Renewable energy is produced by a backstop, while the consumption good is produced from energy – provided by the backstop or from a stock of fossil fuels – and labor. Along the equilibrium path, we show that the stock of fossil fuels might or might not have been completely depleted. Under the first possibility, the forward-looking competitive equilibrium can be computed recursively from the steady state of the economy. This is however no longer possible under the second possibility where the part of the resource stock *left in situ* serves as the *oil bubble*. In this case, long run equilibrium indeterminacy arises with a continuum of possible steady states. Also, the dynamic convergence to a steady state is far from being simply monotone, and might exhibit cyclical behavior, such as damped oscillation, limit cycles, etc.

Keywords: Exhaustible Resources, Endogenous Fertility, Overlapping Generations, Complex Dynamics

JEL Classification: J13, O41, Q30

* Département économique, Université Laval, Cité Universitaire, Ste Foy, Québec, Canada G1K 7P4 ;
nmhung@ecn.ulaval.ca.

** Département de science économique, Université d'Ottawa, 55 Laurier E, Ottawa, Ontario, Canada, K1N 6N5; nvquyen@uottawa.ca.

1. INTRODUCTION

Malthusian stagnation – an explosive population and increasing scarcity of natural resources – has been the motivation behind numerous simulation studies that culminated in the publication by The Club of Rome a popular book entitled “The Limits to Growth,” authored by Meadow et. al. (1972), a team of leading MIT engineers. At first, academic economists objected to the trend projections used to simulate the future, arguing that human beings act rationally in the face of resource scarcity, and will solve the overpopulation problem by making the number of offspring part of the economic decision. Not long after the publication of this unpleasant prophecy, the oil crisis in the beginning of the 70’s rang the alarm, calling for further inquiries on a problem that seems to be real, and, therefore, cannot be disregarded cavalierly.

While the role of exhaustible resources in economic growth has almost then been thoroughly explored (see the well-known Review of Economic Studies 1974 Symposium), the question of endogenous population and fertility decision has only attracted attention – once again – more recently (see Becker and Barro (1988), Barro and Becker (1989), Becker, Murphy, and Tamura (1990), Galor and Weil (1999, 2000), among others). To our knowledge, a synthesis of these two strands of literature has not been undertaken, despite the fact that such a synthesis certainly contributes to our understanding of the real world. This is the task that we propose to accomplish.

In the first strand of literature, most of economic models considered are of the Ramsey-Koopmans type in which the size of the population – assumed to be exogenously given – has seldom been a concern. The basic questions addressed by this strand of the literature are: (i) How does the market allocate an exhaustible resource stock over time, and what is the time path of the resource price? (ii) Is the market efficient in allocating the exhaustible resource over time? and (iii) What are the implications of resource exhaustibility in the context of economic growth? Some fine analyses to this panoply of questions can be found in Dasgupta and Heal (1974, 1979). The answers to the first two questions are that the resource price rises at the rate of return for holding assets – the so-called Hotelling rule – and this would warrant allocation efficiency along the resource extraction path, with the resource being depleted asymptotically. As to the third question,

whether economic growth is sustainable is the main focus. Introducing exogenous neutral technical progress, Stiglitz (1974) found that growth is possible with a stationary depletion policy determined by the saving rate given in the economy, even if the resource input is essential. For the case of Cobb-Douglas production in which the resource factor is essential, but not important (its share in the production is less than the share of labor), a non-degenerate steady state is also possible. In general, without exogenous technical progress, and when the resource is essential in the production process, the economy would sink in the long run to the trivial steady state in which both the resource and the capital stock vanish, and so does the consumption. This unpleasant outcome could only be avoided when the exhaustible resource can be substituted for by a reproducible capital. Also, the existence and characterization of the optimal solution have been fully analyzed – in the setting of an optimal growth model – by Mitra (1980), and more recently by Cass and Mitra (1991) for a wide range of technological possibilities, allowing for unbounded consumption in the long run. For a comprehensive survey of this strand of literature, see Krautkraemer (1998).

With respect to the second strand of the literature on economic growth in which the size of the population is endogenous, only reproducible capital has been considered as a factor of production besides labor. In the class of models that follow the Ramsey-Solow tradition in which all economic decisions are conferred to a single infinitely lived agent (a planner, or the head of a dynasty), the population size tends to a stationary level; see, for example Razin and U Ben-Zion (1975) or Nerlove, Razin, and Sadka (1987). When capital is human – as in Barro and Becker, *op. cit.* – rather than physical, the appropriate model is of the Uzawa-Lucas variety (see Lucas (1988)). Using also the dynastic-utility formulation, these authors showed that the economy exhibits exponential growth at a rate equal to a positive endogenous fertility rate. Works in this direction have been carefully surveyed in Tamura (2000). In the overlapping-generations framework, Samuelson (1975) investigated the optimal size of the population in the long run and pointed out that the incentive to maintain an increasing fertility rate would ultimately lead to an inefficient allocation outcome. Erlich and Lui (1991) further discussed this question, and a concise literature survey was provided in Erlich and Lui (1997).

Surprisingly, there are not many studies bringing together natural resources and population in a comprehensive synthetic model of economic growth. Exceptions are Nerlove, Razin, and Sadka (1986) and Eckstein, Stern, and Wolpin (1988). These papers focused on indestructible land as a production factor. Nerlove et al. relied upon the dynastic-utility approach to show the efficiency of the market outcome with endogenous population, while Eckstein et al. used the overlapping-generations framework, and demonstrated that as long as the fertility decision is taken into account, the population growth will not be excessive; the market outcome will be efficient; and the economy will reach a stationary long run consumption level above the Malthusian subsistence level. The value of land depends on the time path of land per capita and, since land is fixed in quantity, the problem of over-accumulation of capital is simply ruled out. On the other hand, Nerlove (1993) is, to our knowledge, the only study that pieces together the use of a renewable resource and the fertility decision. Studies linking exhaustible resources with fertility decisions are simply non-existent; on this point, see Nerlove and Rault (1997), and Robinson and Srinivasan (1997).

In this paper, we make use of the overlapping generations (OLG) model *à la* Allais-Samuelson-Diamond tradition to formulate and analyze the relationships among exhaustible resources, technology transition, and endogenous fertility (see Geneakoupoulos and Polemarchakis (1991) for the present state of the art). Four classes of economic agents co-exist in each period: a young generation, an old generation, competitive firms producing a consumption good, and competitive firms producing renewable energy. Renewable energy is produced by a backstop from capital, say solar collectors, while the consumption good is produced using energy – provided by the backstop or from a stock of fossil fuels – and labor. The consumption good can also be used as investment goods to augment the stock of backstop capital. While oil can be extracted at negligible cost, its ultimate stock is limited. The backstop, on the other hand, can provide a perpetual flow of energy. However, energy produced by the backstop requires capital, and as the stock of fossil fuels dwindles, it is imperative that backstop capital be accumulated to avoid a drastic cut in consumption. Thus the transition from oil to backstop becomes an interesting question.

Economic agents interact on five markets – oil, solar energy, labor, capital, and the consumption good. The two real assets in the economy – capital and oil – represent the only possible forms of saving, and in any period they are owned by the old generation of that period. An individual works when she is young, and retires when she is old. She has to allocate her wages among current consumption, raising children, and saving for old-age consumption. The lifetime utility of a young individual depends on her current consumption, her old-age consumption, and the number of offspring she raises, and these variables are assumed to be separable in the lifetime utility function. Furthermore, future utilities of consumption are discounted, and the single-period sub-utility function of consumption is assumed to be concave, strictly increasing, and satisfy the Inada conditions, i.e. the marginal utility of consumption tends to infinity (zero) when consumption tends to zero (infinity). As for the sub-utility function of offspring, it is assumed to be concave, and is strictly increasing as the number of offspring rises from 0 to a saturation level. However, unlike the single-period sub-utility function of consumption, the marginal utility of offspring is assumed to be finite when the number of offspring is 0. It is this assumption that gives our model a Malthusian flavor: when wages are low, the birthrate will be almost 0.

The literature on exhaustible resources under the OLG framework is rather sparse, maybe because studying thoroughly the extraction of exhaustible resources in this framework is too involved. Kemp and Long (1979) and, lately, Olson and Knapp (1997) are exceptions, Kemp and Long assumed that the resource is not essential in the production process, and showed that in steady state the resource could be partially depleted, inducing, therefore, a form of inefficiency in this case. On the other hand, Olson and Knapp considered the exhaustible resource as an essential factor of production. They established the existence of an equilibrium and provided a characterization of the market outcome. Market efficiency in this study is warranted; however, the economy would contract and ultimately collapse into the trivial steady state of zero output in the limit. The convergence to this degenerate state need not be monotone, but may happen in damped oscillations, and the pattern of resource extraction as well as the time path of the resource price could possibly exhibit non-classical behavior. Quite recently, Agnani et al. (2003) extended the Olson and Knapp model by introducing exogenous technological

progress and producible capital as a factor of production together with an exhaustible resource. With logarithmic additive utility and Cobb-Douglas production, they found that the economy exhibits a positive steady-state growth rate if the labor share is high enough, and this balanced growth path is efficient.

Notwithstanding the afore-mentioned works, we impose no specific functional form in our analysis. In our model, children are treated as a consumption good, and the number of offspring in each period is the result of the lifetime utility maximization of the young generation of that period. Our formulation thus stands in contrast with the dynastic formulations of Becker and Barro (1988) and Barro and Becker (1989) in which the utility of a parent depends on her own consumption, the number of offspring she raises, and the total utilities of all her offspring, and in which the decisions on fertility are made at the beginning of time by the head of the dynasty. Unlike the dynastic-utility formulation, which assumes perfect foresight on the part of the head of the dynasty and to whom the task of inter-temporal resource planning is assigned, the overlapping-generations model provides a decentralized setting. Our model is thus somewhat more market oriented than the central-planning view; the latter requires perfect foresight on the part of the head of the dynasty and a binding contract across generations. In addition to the fact that under the OLG framework the society is composed of mortal individuals who can trade through their lifetime, and with resource assets which may act as stores of values for saving purpose, we claim that our modeling strategy is more appropriate. At least, it does not go against the strong empirical evidence (see J. G. Atonji et.al. (1992)) which does not support the hypothesis that members of extended families are altruistically linked in the way presumed by the dynastic type model.

As we have indicated earlier, if there is no possibility of substitution for an essential exhaustible resource, say oil, used as an input in the production process, the economy might glide to the trivial steady state of zero consumption in the long run, a regretful doomsday. In order to reach the state of economic sustainability, the problem of technology transition emphasizes the possibility of substituting for the exhaustible resource with an everlasting source of energy input, say solar energy, which could be made available through investments in the so-called backstop technology (see, for example, Hung and Quyen (1993, 1994) for partial equilibrium analyses, and Tahvonon

and Salo (2001) in the context of economic growth). In these models, oil is used first, with solar energy gradually being brought in to substitute for oil. By the time the oil stock is completely depleted, the backstop capital will have reached the Golden Rule level, and its marginal productivity is equal to the interest rate. Can these results be carried over into a dynamic general-equilibrium framework, especially when the population is not a datum, but endogenous in the sense that it results from fertility decisions made by economic agents?

The introduction of endogenous fertility into an OLG model of exhaustible resources and economic growth changes the nature of the problem of resource depletion in two fundamental ways. First, it is not the resource scarcity in absolute terms, but the resource scarcity per capita that matters. Although the absolute size of the remaining resource stock necessarily declines through time due to extraction, the resource endowment per worker might rise or decline through time, depending on the birthrates chosen by the successive young generations, and this leads to a more general Hotelling rule which is related to the birth rate (the biological interest rate). Second, the temporary equilibria can no longer be computed recursively, and the existence of a competitive equilibrium becomes problematic, especially because the market size becomes endogenous with agent's fertility decision. Furthermore, a birthrate that is arbitrarily close to 0 in one period leads, in the next period, to an abundance of resource or capital endowment per capita, and this means that many limiting arguments must be deployed to prevent the population from collapsing in finite time.

The mechanism that operates in our model can be described as follows. In any period, if the resource scarcity per capita – the endowments of oil plus capital per worker – are high, the energy input per worker will be high, leading to a high wage rate. The high wage rate in turn induces a high level of current consumption, a high level of future consumption, and more importantly, a number of offspring close to the saturation level – assuming that the cost in terms of real resources of raising a child is constant. The endowments per worker in the next period, although still high, will be lower than that of the current period. On the other hand, when the resources per capita are scarce, wages will be low, with the ensuing consequences that current consumption, future consumption, and fertility will also be low. In particular, if wages are extremely low,

most of the wages will be devoted to consumption, and fertility will be extremely low. This last result follows from the assumption that the Inada conditions are imposed on the single-period sub-utility function of consumption, but not on the sub-utility function of offspring. Thus when the wage rate descends to a critical level, fertility will decline to 0, leading to an abundance of resources per capita in the next period. An abundance of resources per capita in the next period, as already argued, leads to a high fertility rate in that period, and allows the population to bounce back.

When fossil fuels are abundant at the beginning, a competitive equilibrium in our model consists of three phases. In the first phase, the energy inputs used in the production of the consumption good come solely from oil. If the population is stable or growing in the long run, oil alone cannot sustain the economy indefinitely, and the backstop must be brought into use at some time to provide part of the energy requirements of the economy. The time interval that encompasses the introduction of the backstop and the end of extraction activities constitutes the second phase of a competitive equilibrium: the phase of technology transition. The third phase of a competitive equilibrium begins after all extraction activities have been terminated, either due to oil exhaustion or because the competitive equilibrium in question involves incomplete oil depletion.

Incomplete oil depletion merits some elaboration. In traditional models of resource extraction – à la Hotelling or in the tradition of optimal growth – the resource is always presumed to be completely depleted, and the equilibrium to be efficient; the only concern is to see how the resource is exploited through time. In an OLG model, this presumption turns out to be unfounded. If the *time horizon is finite*, then any amount of oil that remains at the beginning of the last period will be utilized as part of the energy input used in the production of the consumption good. Furthermore, an economy with a finite time horizon is an Arrow-Debreu economy, and a competitive equilibrium of such an economy is Pareto optimal according to the first theorem of welfare economics. However, when the *time horizon is infinite*, and agents make plans only for the two periods of their life cycles, we cannot take for granted that the resource stock will be completely depleted or that the equilibrium will be efficient. In her lifetime plan, an agent does not consider the impact of her decision on the welfare of future generations. Furthermore, when the time horizon is infinite, a young individual can always invest in

oil, and then sells the oil she owns to the young generation or the competitive firms of the next period. The fact that the resource is exhaustible has no bearing on her investment decision, and as long as the price of the resource rises through time at the market rate of interest, she is quite willing to hold this asset so that the resource stock will only be partially depleted. The part of the resource stock left in situ unexploited serves as a *financial bubble*, and in steady state the resource price – according our modified version of the Hotelling rule – will rise geometrically through time at a rate equal to the marginal product of capital, which is also the steady-state birthrate.

Besides incomplete depletion, we should also mention that, unlike the planning dynastic framework in which one sources of energy are used sequentially, oil and backstop might be used simultaneously in the production process under the OLG framework. Also, it is remarkable, but not surprising, that the introduction of oil into the model gives rise to multiple equilibria with complex dynamics, which includes the possibility of convergence to a steady state through damped oscillation, limit cycles, etc.

The paper is organized as follows. In Section 2, the overlapping-generations model is presented. In Section 3, we study preliminarily the competitive equilibrium of an economy which does not have any oil left, and which is endowed only with backstop capital. We demonstrate the existence of a unique forward-looking temporary equilibrium as well as the existence of at least a steady state under infinite time horizon. We then show the possibility of oscillation and of a 2-cycle in the dynamic convergence to a steady state. To support all these findings, we provide a numerical example for each case. In Section 4, we focus on an economy endowed with both oil and backstop capital and state the existence of a competitive equilibrium under infinite horizon. In Section 5, we characterize the equilibrium oil extraction path, and provide the conditions under which the oil stock will be exhausted in finite time. In Section 6, we discuss the possibility of incomplete oil depletion, and show that there might exist infinitely many steady states in which the oil stock is only partially depleted. Furthermore, we show that if the rate of capital depreciation is low, then in addition to the equilibrium with incomplete exhaustion there also exists an equilibrium in which the oil stock is depleted in finite time. Thus there might exist multiple equilibria. Moreover, in the long run, the birth rate is lower under incomplete than under complete oil exhaustion. In Section 7, we bring

together all the disparate elements into a synthetic characterization of the competitive equilibria that emerge from our model. Section 8 contains a summary of our major findings and some concluding remarks.

2. THE MODEL

2.1. *The Technology*

The perfectly competitive firms produce a consumption good from two inputs – labor and energy – according to a standard neoclassical production function, say $Y = F(E, L)$, where Y denotes the output; E the energy input; and L the labor input. In what follows, we shall let $e = E/L$ denote the energy input per worker, and $f(e) = F(e, 1)$ denote the output of the consumption good produced by a worker. We assume that $f'(e) > 0$ for all $e > 0$, $\lim_{e \rightarrow 0} f'(e) = \infty$ when $e \rightarrow 0$, and $\lim_{e \rightarrow \infty} f'(e) = 0$ when $e \rightarrow \infty$.

In our economy, energy inputs come from two sources: oil and a backstop, say solar energy. While oil can be extracted at negligible cost, its ultimate stock is limited. The backstop, on the other hand, can provide a perpetual flow of energy. However, harnessing the Sun's energy requires investments in backstop capital, say solar collectors. In any period, the amount of solar energy harnessed is assumed to be proportional to the stock of backstop capital K , and, to simplify the exposition, we shall assume that the proportionality constant is equal to unity, i.e., one unit of backstop capital produces one Btu. Also, we shall assume that backstop capital depreciates at rate δ , $0 \leq \delta \leq 1$.

If Q_t is the amount of oil – also measured in Btu's – extracted for use as part of the energy input in period t , and K_t is the stock of backstop capital in that period, then the total energy input used in period t is $E_t = Q_t + K_t$. Furthermore, if L_t is the labor input used in period t , then the output of the consumption good in that period is $Y_t = F(Q_t + K_t, L_t)$. We assume that the consumption good can also be used as investment goods to augment the stock of backstop capital. As time goes on, and the oil resources dwindle, it is imperative that investments in the backstop be made to prevent a drastic reduction in consumption. The accumulation of backstop capital only influences the output of the consumption good indirectly through the amount of solar energy delivered by the backstop sector to the economy. Because there is only one kind of

capital in the model, namely backstop capital, we shall from now on refer to backstop capital simply as capital.

2.2. Economic Agents

In the economy, four classes of economic agents coexist in each period: a young generation, an old generation, competitive firms producing the consumption good, and competitive firms producing solar energy. These economic agents interact on five markets – oil, solar energy, labor, backstop capital, and the consumption good. An individual works when she is young. She allocates her wages among current consumption, raising children, and saving for her old-age consumption. The two real assets in the economy are oil and capital, which represent the only possible forms of saving.

At the beginning of each period $t, t = 0, 1, \dots$, the state of the economy is represented by a list (X_t, K_t, N_t^0, N_t^1) , where X_t, K_t, N_t^0 , and N_t^1 represent, respectively, the remaining oil stock, the stock of capital, the number of young individuals, and the number of old individuals. The initial state of the economy, i.e., (X_0, K_0, N_0^0, N_0^1) , is assumed to be known. The consumption good in each period is taken as the numéraire, and for each $t = 0, 1, \dots$, let $\phi_t, \varphi_t, \omega_t$, and ρ_t denote, respectively, the price of oil, the price of solar energy, the wage rate, the rental rate of backstop capital – all in period t .

2.2.1. The Old Generation

The real assets in each period are owned by the old generation of that period. An old individual in period t owns X_t / N_t^1 units of oil and K_t / N_t^1 units of backstop capital, and her consumption is thus given by $c_t^1 = [\phi_t X_t + (1 - \delta + \rho_t) K_t] / N_t^1$.

2.2.2. The Young Generation

A young individual owns nothing except for 1 unit of labor that she supplies inelastically on the labor market. She has to allocate her labor income among current consumption, raising children, and saving for old-age consumption. A lifetime plan for a young individual of period t is a list $(c_t^0, c_{t+1}^1, b_t, x_{t+1}, k_{t+1})$, where $c_t^0, c_{t+1}^1, b_t, x_{t+1}$, and k_{t+1} denote, respectively, her current consumption, her old-age consumption, the number of offspring she raises – at the constant cost h in terms of real resources per child – the

amount of oil she buys as investment, and the amount of capital she buys – also for investment purposes. The lifetime utility associated with such a lifetime plan is assumed to be given by

$$(1) \quad u(c_t^0) + \gamma u(c_{t+1}^1) + v(b_t),$$

where $u(c)$ is the single-period sub-utility function associated with consumption, and $v(b)$ is the sub-utility function of offspring. Also, $\gamma, 0 < \gamma < 1$, is a parameter representing the factor she uses to discount future utilities. It should be emphasized that for parents children have intrinsic value, and the number of offspring is here considered as a consumption good from their viewpoint.¹ We impose the following assumption on the single-period sub-utility function of consumption and the sub-utility function of offspring:

ASSUMPTION 1: *The sub-utility function of consumption $u(c)$ is defined for all $c > 0$. It is continuously differentiable, strictly concave, and strictly increasing. Furthermore, it satisfies the following Inada conditions: $\lim_{c \rightarrow 0} u'(c) = +\infty$ and $\lim_{c \rightarrow +\infty} u'(c) = 0$. As for the sub-utility function of offspring, $v(b)$ is defined for all $b \geq 0$. It is continuously differentiable and concave. Furthermore, there exists a number $b^{\max} > 1$ such that $v'(b) > 0$, $0 \leq b < b^{\max}$, and $v'(b) \leq 0, b \geq b^{\max}$.*

Note that b^{\max} represents the saturation number of offspring. Because the single-period sub-utility function of consumption and the sub-utility function of offspring² are both

¹ One may think of b_t as a measure of quality of a child – in period t – for an economy in an economy endowed with a constant population, say N_0 . Then the economy's human capital at the beginning is $N_0^0 b_0$, and the economy's human capital in the following periods are given by $N_0 b_t, t = 1, 2, \dots$ We think that when one talks about human capital, the investments involved should encompass both the efforts made with regard to the quantity of children and the efforts made with regard to the quality of each child. Here, the quantity-quality trade-off is relevant and merits a separate study. In this paper, we try not to be abusive by engaging in a lax interpretation of human capital. Therefore, we choose to consider b as the number of offspring, and thus $N_t^0 = N_0^0 b_0 \dots b_{t-1}$ is the size of the young generation in period t . The size of the young generation in each period is endogenous precisely because b_t is a decision variable for a young individual in period t .

² Observe that part (ii) of Assumption 1 rules out homothetic preferences. If preferences are homothetic, the Engel curves of current consumption, future consumption, and offspring are all straight lines. In particular, when labor income rises, the demand for offspring rises in the same proportion as the rise in labor income,

assumed to be concave and increasing, current consumption, old-age consumption, and offspring are all normal goods. Furthermore, because it is costly to raise children, the optimal number of children is strictly less than b^{\max} .

The problem of a young individual in period t is to find a feasible lifetime plan that maximizes (1) subject to the inter-temporal budget constraint

$$(2) \quad c_t^0 + c_{t+1}^1 / r_{t+1} + hb_t - \omega_t = 0,$$

where we have let

$$(3) \quad r_t = \max\{\phi_t / \phi_{t-1}, 1 - \delta + \rho_t\}$$

denote the interest rates she earns on her savings.

Note that the Inada condition imposed on the sub-utility function associated with consumption implies that $c_t^0 > 0$ and $c_{t+1}^1 > 0$; however, the number of offspring raised by a young individual of period t might be zero if the current wage rate is low enough. The saving of the individual is

$$(4) \quad s_t = \omega_t - c_t^0 - hb_t,$$

and the division of saving between oil and capital depends on their relative rates of return. If $\phi_{t+1} / \phi_t > 1 - \delta + \rho_{t+1}$, then $x_{t+1} = s_t / \phi_t$ and $k_{t+1} = 0$. If $\phi_{t+1} / \phi_t < 1 - \delta + \rho_{t+1}$, then $x_{t+1} = 0$ and $k_{t+1} = s_t$. When $\phi_{t+1} / \phi_t = 1 - \delta + \rho_{t+1}$, the individual is indifferent between oil and capital, and k_{t+1} can assume any value between 0 and s_t .

What happens to the optimal lifetime plan (c_t^0, c_{t+1}^1, b_t) when the rate of return to saving in the next period rises? Because a rise in r_{t+1} makes the price of 1 unit of old-age consumption cheaper, we expect that the substitution effect will cause old-age consumption to rise at the expense of current consumption and the number of offspring. Furthermore, as r_{t+1} rises, real lifetime income also rises with r_{t+1} . The income effect raises current consumption, old-age consumption, and the number of offspring. The income effect reinforces the substitution effect, and causes old-age consumption to rise

which seems to be untenable. Furthermore, for an economy that is sustained only by renewable energy resources, homothetic preferences imply that from any initial condition the economy enters a steady state after one period: the transition to its steady state level of the birth rate lasts exactly one period, and this also seems unreasonable. On the other hand, it can be shown that homothetic preferences allow for a much simpler proof of the existence of a competitive equilibrium for the case the economy begins with a positive stock of fossil fuels.

even more.. However, for current consumption and the number of offspring, the net impact is ambiguous because the substitution effect and the income effect operate in opposite directions. The net impact on s_t is thus ambiguous although c_{t+1}^1 is increasing in r_{t+1} . To obtain sharper results, we shall make the following assumption:

ASSUMPTION 2: For a young individual, current consumption, old-age consumption, and offspring are gross substitutes

Assumption 2 is often made in overlapping-generations models and looks quite innocuous at the macroeconomic level; see, for example Azariadis (1993, Section 7.4). It follows from this assumption that the current consumption of a young individual and the number of offspring she raises will decline when the discounted price of future consumption declines. Hence saving is an increasing function of the rate of interest. On the other hand, for a young individual, current consumption, old-age consumption, and offspring are all normal goods. Thus we expect c_t^0 , c_{t+1}^1 , and b_t to rise with ω_t . Furthermore, because $c_{t+1}^1 = r_{t+1}s_t$, saving also rises with ω_t .

Now according to Assumption 1, the Inada condition is imposed on the sub-utility function of consumption, but not on the sub-utility function of offspring. Thus we can expect that when the labor income of a young individual is too low, she will choose not to raise children. To determine the critical level of labor income that triggers the extinction of the population at the end of the following period, let

$$(5) \quad \omega^{\min}(\delta) = \text{Inf} \{ \omega_t | b_t > 0, \text{ given that } r_{t+1} = 1 - \delta \}$$

As defined, $\omega^{\min}(\delta)$ is the critical wage rate at or below which a young individual will choose not to raise children, given that the rate of return to saving is equal to $1 - \delta$, its minimum possible level. Furthermore, if $\omega^{\min}(\delta)$ is the labor income of a young individual, then using Assumption 2, we can assert that for any rate of return above the minimum level $1 - \delta$, the individual still chooses not to raise children. Let $e^{\min}(\delta)$ denote the energy input per worker that gives rise to the critical wage rate $\omega^{\min}(\delta)$. These two critical variables are linked by the relation: $\omega^{\min}(\delta) = f(e^{\min}(\delta)) - e^{\min}(\delta)f'(e^{\min}(\delta))$. As $\omega_t \downarrow \omega^{\min}(\delta)$, the number of offspring she raises will tend to 0 while the saving for old-

age consumption is bounded below and away from 0, which implies that $s(\omega_t, r_{t+1})/b(\omega_t, r_{t+1})$, the saving/offspring ratio, will tend to infinity. It is this property that prevents the population from collapsing in finite time. When the labor income declines to the critical level $\omega^{\min}(\delta)$, the birthrate approaches 0, but the saving for old-age consumption – although low – is still bounded below and away from 0, allowing for a high saving/offspring ratio. The high saving/offspring ratio means a high level of energy input per worker in the next period, with an ensuing high wage rate in that period. A high wage rate in the next period leads to a high birthrate in that period, which gives the population a chance to bounce back. The saving/offspring ratio also tends to infinity when labor income tends to infinity. The reason is that the birthrate, although rises with income, remains bounded above by the saturation level b^{\max} while saving increases without bound. Thus when the wage rate is high, the cost of raising children becomes a negligible fraction of labor income, and most of the labor income is spent on current consumption and on investments to provide for old-age consumption. Thus, we can expect the curve $\omega_t \rightarrow s(\omega_t, r_{t+1})/b(\omega_t, r_{t+1})$, $\omega_t > \omega^{\min}(\delta)$, to have a U-shape.

2.2.3. Solar Energy Producers

Solar energy is produced by competitive firms from capital. The representative solar energy producer solves the following profit maximization problem:

$$\max_{(K^\#, S^\#)} [\varphi_t S^\# - \rho_t K^\#]$$

subject to the technological constraint $S^\# - K^\# \leq 0$, where $K^\#$ and $S^\#$ represent, respectively, this firm's demand for capital and its output of solar energy.

2.2.4. Producers of the Consumption Good

In each period t , the representative firm in the consumption good sector solves the following profit maximization problem:

$$\max_{(Q, S, L, Y)} [Y - \phi_t Q - \varphi_t S - \omega_t L]$$

subject to the technological constraint $Y - F(Q + S, L) \leq 0$, where Q , S , L , and Y represent, respectively, the oil input, the solar energy input, the labor input, and the output of the consumption good. Let (Q_t, S_t, L_t, Y_t) be a solution of the preceding profit maximization problem. We have (i) $Q_t = 0$ if $\phi_t > \varphi_t$, (ii) $S_t = 0$ if $\phi_t < \varphi_t$. When

$\phi_t = \varphi_t$, the mix (Q_t, S_t) is indeterminate, although the sum $E_t = Q_t + S_t$ is uniquely determined.

2.3. Definition of Competitive Equilibrium

Let $\mathbf{P} = (\phi_t, \varphi_t, \omega_t, \rho_t)_{t=0}^{\infty}$ be a price system. An allocation induced by \mathbf{P} is a list of infinite sequences

$$\mathbf{A} = (c_0^1, (c_t^0, c_{t+1}^1, b_t, x_{t+1}, k_{t+1})_{t=0}^{\infty}, (Q_t, S_t, L_t, Y_t)_{t=0}^{\infty}, (K_t^{\#}, S_t^{\#})_{t=0}^{\infty}, (X_t, K_t, N_t^0, N_t^1)_{t=0}^{\infty})$$

with the following properties: Under the price system \mathbf{P} ,

- (i) $c_0^1 = [\phi_0 X_0 + (1 - \delta + \rho_0) K_0] / N_0^1$;
- (ii) $(c_t^0, c_{t+1}^1, b_t, x_{t+1}, k_{t+1})$ is the optimal lifetime plan for a young individual of period t ;
- (iii) $(K_t^{\#}, S_t^{\#})$ is an optimal production plan of the representative firm in the backstop sector in period t ;
- (iv) (Q_t, S_t, L_t, Y_t) is an optimal production plan of the representative firm in the consumption good sector in period t ;
- (v) $(X_t, K_t, N_t^0, N_t^1) = N_{t-1}^0(x_t, k_t, b_{t-1}, 1)$.

The pair (\mathbf{P}, \mathbf{A}) is said to constitute a *competitive equilibrium* if the following market-clearing conditions are satisfied for each $t = 0, 1, \dots$,

- (vi) $X_{t+1} + Q_t = X_t$,
- (vii) $S_t = S_t^{\#}$,
- (viii) $L_t = N_t^0$,
- (ix) $K_t^{\#} = K_t$,
- (x) $N_t^1 c_t^1 + N_t^0 (c_t^0 + h b_t + k_{t+1}) = Y_t + (1 - \delta) K_t$.

3. COMPETITIVE EQUILIBRIUM FOR AN ECONOMY WITHOUT OIL RESOURCES

Suppose that the economy begins in state (K_0, N_0^0, N_0^1) in period 0, with $K_0 > 0$, $N_0^0 > 0$, and $N_0^1 > 0$; that is, the economy begins without any oil resources, but with a positive stock of capital and a positive population. Because one unit of capital

produces one Btu, and because in equilibrium the profit in the backstop sector is 0, the price of renewable energy is equal to the rental rate of capital. Thus, we shall conduct our analysis in terms of the rental rate of capital without explicitly mentioning the price of renewable energy.

3.1. *The Capital/Labor Ratio*

When there are no oil resources, all the energy needs of the economy are provided by the backstop. To prevent the population from becoming extinct in period 1, we shall assume that $\kappa_0 > e^{\min}(\delta)$, where we have let $\kappa_0 = K_0 / N_0^0$.

In period 0, the equilibrium rental rate of capital and the equilibrium wage rate are given, respectively, by $\rho_0 = f'(\kappa_0)$ and $\omega_0 = f(\kappa_0) - \kappa_0 f'(\kappa_0)$. The consumption of an old individual in period 0 is given by $c_0^1 = (1 - \delta + \rho_0)K_0 / N_0^1$. As for a young individual of period 0, the current consumption, the old-age consumption, the number of offspring, and the saving under the form of capital that constitute the solution of her lifetime utility maximization problem are denoted by $c^0(\omega_0, 1 - \delta + \rho_1)$, $c^1(\omega_0, 1 - \delta + \rho_1)$, $b(\omega_0, 1 - \delta + \rho_1)$, and $k(\omega_0, 1 - \delta + \rho_1)$, respectively. The capital labor ratio in period 1 that is generated by this optimal lifetime plan is

$$\kappa(\omega_0, 1 - \delta + \rho_1) = k(\omega_0, 1 - \delta + \rho_1) / b(\omega_0, 1 - \delta + \rho_1).$$

3.2. *Existence and Uniqueness of Competitive Equilibrium for an Economy without Oil Resources*

For any wage rate $\omega_0 > \omega^{\min}(\delta)$ and any rental rate of capital $\rho_1 \geq 0$ in period 1, let

$$\zeta(\omega_0, \rho_1) = f'(\kappa(\omega_0, 1 - \delta + \rho_1)).$$

As defined, $\zeta(\omega_0, \rho_1)$ represents the rental rate of capital in period 1 generated by the maximizing behavior of a young individual of period 0, given that ω_0 is her labor income and ρ_1 is the rental rate of backstop capital that this individual expects to prevail in period 1. It is clear that the curve $\zeta(\omega_0, \cdot)$ is continuous. Because a young individual must save for her old-age consumption, her capital investment is always positive even if its rate of return is 0; that is, $0 < k(\omega_0, 1 - \delta) < \omega_0$. Furthermore, because $\omega_0 > \omega^{\min}(\delta)$, we must also have $b(\omega_0, 1 - \delta) > 0$. Hence $\kappa(\omega_0, 1 - \delta) > 0$, which implies

$0 < \zeta(\omega_0, 0) < \infty$. Also, according to Assumption 2, the capital labor ratio $\kappa(\omega_0, 1 - \delta + \rho_1)$ is increasing in ρ_1 , which implies that $\zeta(\omega_0, \cdot)$ is downward sloping. Now as ρ_1 continues to rise, if $b(\omega_0, 1 - \delta + \rho_1) = 0$ for some value $\rho_1 = \tilde{\rho}_1$, then $\kappa(\omega_0, 1 - \delta + \rho_1) \rightarrow +\infty$ when $\rho_1 \uparrow \tilde{\rho}_1$, which means that $\zeta(\omega_0, \rho_1) \downarrow 0$ as $\rho_1 \uparrow \tilde{\rho}_1$, and $\zeta(\omega_0, \cdot)$ must have crossed the 45-degree line before ρ_1 reaches $\tilde{\rho}_1$. On the other hand, if $b(\omega_0, \rho_1) > 0$ for all $\rho_1 \geq 0$, then $\zeta(\omega_0, \cdot)$ must also cross the 45-degree line at a single point. In either case, $\zeta(\omega_0, \cdot)$ crosses the 45-degree line at a single point, which represents the equilibrium rental rate of capital in period 1. Note that at this equilibrium the birthrate is positive. We have just established the following lemma:

LEMMA 1: *Suppose that ω_0 , the wage rate prevailing in period 0, is strictly above the critical level $\omega^{\min}(\delta)$. There exists a unique value for the rental rate of capital in period 1 that satisfies the condition $f'(\kappa(\omega_0, 1 - \delta + \rho_1)) = \rho_1$. The unique value of ρ_1 , say $\rho_1 = g(\omega_0)$, that satisfies this condition is the equilibrium rental rate of capital in period 1, given that ω_0 is the wage rate prevailing in period 0. Furthermore, the equilibrium birthrate in period 0 is strictly positive.*

In period 1, the equilibrium capital labor ratio, the equilibrium price of solar energy, and the equilibrium wage rate are given by $\kappa_1 = \kappa(\omega_0, 1 - \delta + \rho_1) = [f']^{-1}(\rho_1)$, $\varphi_1 = \rho_1$, and $\omega_1 = f(\kappa_1) - \kappa_1 f'(\kappa_1) > \omega^{\min}(\delta)$, respectively. Also, the state of the system in period 1 is $(K_1, N_1^0, N_1^1) = (N_0^0 k(\omega_0, 1 - \delta + \rho_1), N_0^0 b(\omega_0, 1 - \delta + \rho_1), N_0^0)$. The procedure used to obtain $(\varphi_1, \omega_1, \rho_1)$,

$$(c^0(\omega_0, 1 - \delta + \rho_1), c^1(\omega_0, 1 - \delta + \rho_1), b(\omega_0, 1 - \delta + \rho_1), k(\omega_0, 1 - \delta + \rho_1)),$$

and (K_1, N_1^0, N_1^1) can be repeated ad infinitum to obtain a price system $\mathbf{P} = (\varphi_t, \omega_t, \rho_t)_{t=0}^{\infty}$ and an allocation induced by \mathbf{P} , say

$$\mathbf{A} = (c_0^1, (c_t^0, c_{t+1}^1, b_t, k_{t+1})_{t=0}^{\infty}, (S_t, L_t, Y_t)_{t=0}^{\infty}, (K_t^{\#}, S_t^{\#})_{t=0}^{\infty}, (K_t, N_t^0, N_t^1)_{t=0}^{\infty}),$$

where

$$\begin{aligned} & (c_t^0, c_{t+1}^1, b_t, k_{t+1}) \\ & = (c^0(\omega_t, 1 - \delta + \rho_{t+1}), c^1(\omega_t, 1 - \delta + \rho_{t+1}), b(\omega_t, 1 - \delta + \rho_{t+1}), k(\omega_t, 1 - \delta + \rho_{t+1})), \end{aligned}$$

and $S_t = S_t^\# = K_t^\# = K_t, L_t = N_t^0, Y_t = F(S_t, L_t)$. The pair (\mathbf{P}, \mathbf{A}) , thus constructed, constitutes the unique competitive equilibrium for an economy without oil resources. We summarize the result just obtained in the following proposition, which bears resemblance to Theorem 13.1, found in Azariadis, op.cit, page 108.

PROPOSITION 1: *For an economy that has no oil resources, but that is sustained by a source of renewable energy – provided by a backstop technology – there exists a unique competitive equilibrium.*

3.3. Steady States for an Economy without Oil Resources: Existence and Uniqueness

Let

$$(6) \quad \rho^{\max}(\delta) = f'(e^{\min}(\delta))$$

denote the critical price of energy at or above which a young individual will choose not to raise children. Now for any initial rental rate of capital ρ_0 that satisfies the condition $0 < \rho_0 < \rho^{\max}$, define

$$(7) \quad G(\rho_0) = g(\omega(\rho_0)),$$

where we have let $\omega(\rho_0)$ denote the prevailing equilibrium wage rate when ρ_0 is the equilibrium rental rate of capital; that is, $\omega(\rho_0) = f(\kappa_0) - \rho_0 \kappa_0$, with $\kappa_0 = [f']^{-1}(\rho_0)$.

The map $G: \rho_0 \rightarrow G(\rho_0), 0 < \rho_0 < \rho^{\max}(\delta)$, plays a fundamental role in our analysis. It describes the transition of the equilibrium rental rate of capital from one period to another for an economy without oil or for an economy that has exhausted its oil resources. A fixed point of G represents a steady-state level for the rental rate of capital. The following lemma presents some of the limiting behavior of G .

LEMMA 2: *We have $\lim_{\rho_0 \uparrow \rho^{\max}(\delta)} G(\rho_0) = 0$ and $\lim_{\rho_0 \downarrow 0} G(\rho_0) = 0$. Also, $G'(\rho_0) > 1$. for all ρ_0 in a right neighborhood of 0.*

PROOF: First, note that $\kappa(\omega(\rho_0), 1 - \delta) \rightarrow \infty$ when $\rho_0 \uparrow \rho^{\max}(\delta)$. Furthermore, according to Assumption 2, we have

$$\kappa(\omega(\rho_0), 1 - \delta) < \kappa(\omega(\rho_0), 1 - \delta + g(\omega(\rho_0))),$$

which implies that

$$f'(\kappa[\omega(\rho_0), 1 - \delta]) > f'(\omega(\rho_0), 1 - \delta + g(\omega(\rho_0))) = G(\rho_0).$$

Hence

$$\begin{aligned} \lim_{\rho_0 \uparrow \rho^{\max}(\delta)} G(\rho_0) &= \lim_{\rho_0 \uparrow \rho^{\max}(\delta)} f'(\omega(\rho_0), 1 - \delta + g(\omega(\rho_0))) \\ &\leq \lim_{\rho_0 \uparrow \rho^{\max}(\delta)} f'(\kappa[\omega(\rho_0), 1 - \delta]) = 0. \end{aligned}$$

Next, note that as $\rho_0 \downarrow 0$, the capital labor ratio and the wage rate associated with ρ_0 , namely κ_0 , and $\omega(\rho_0)$, both tend to infinity. Because the current wage rate tends to infinity, the current consumption and the future consumption of a young individual both tend to infinity – even when the rental rate of capital in the next period is 0. Also, the number of offspring raised by a young individual will rise to the saturation level b^{\max} . Hence the capital labor ratio generated by the maximizing behavior of a young individual of period 0 will tend to infinity, which implies that the rental rate capital in period 1, namely $G(\rho_0)$, will tend to 0. Finally, note that as $\rho_0 \downarrow 0$, the capital labor ratio generated by the maximizing behavior of a young individual, say κ_1 , satisfies the following inequality: $\kappa_1 < f(\kappa_0) / b^{\max} = [f(\kappa_0) / \kappa_0][\kappa_0 / b^{\max}]$. Due to the Inada condition on f , the expression inside the first pair of square brackets tends to 0 as $\kappa_0 \rightarrow +\infty$. Hence κ_1 / κ_0 is arbitrarily small when ρ_0 is sufficiently small, which means that $G(\rho_0) = \rho_1 = f'(\kappa_1) > f'(\kappa_0) = \rho_0$ for all ρ_0 in a right neighborhood of 0. ■

We shall extend the curve $G: \rho_0 \rightarrow G(\rho_0), 0 < \rho_0 < \rho^{\max}(\delta)$, to all of the closed interval $[0, \rho^{\max}(\delta)]$ by setting $G(0) = 0$ and $G(\rho^{\max}(\delta)) = 0$. Let

$$(8) \quad G^{\max}(\delta) = \max_{0 \leq \rho_0 \leq \rho^{\max}(\delta)} G(\rho_0).$$

Because and $G(0) = G(\rho^{\max}(\delta)) = 0$ and $G(\rho_0) > 0$ for all $\rho_0 \in (0, \rho^{\max}(\delta))$, we must have $G^{\max}(\delta) > 0$. To preclude the possibility that the population becomes extinct in finite time, we shall make the following assumption:

ASSUMPTION 3: We have $G^{\max}(\delta) < \rho^{\max}(\delta)$.

Assumption 3 implies that for an economy sustained completely by renewable energy, if the rental rate of capital is currently below the critical level $\rho^{\max}(\delta)$, it will remain below $\rho^{\max}(\delta)$ in the next period. If $f'(\kappa_0) > G^{\max}(\delta)$, then the equilibrium rental rate of capital will enter the interval $[0, G^{\max}(\delta)]$, called a *confining set*, in period 1, and will never leave the interval after that. Now as ρ_0 rises from 0 to $\rho^{\max}(\delta)$, the curve G rises from the origin and stays above the 45-degree line initially. It reaches the maximum value $G^{\max}(\delta)$ at some point inside the open interval $(0, \rho^{\max}(\delta))$, then descends to the point $\rho^{\max}(\delta)$ on the horizontal axis when ρ_0 reaches $\rho^{\max}(\delta)$. Hence it must cross the 45-degree line at least once, and the rental rate of capital at such a crossing represents the rental rate of capital in a steady state. We have just established the following proposition:

PROPOSITION 2: For an economy that has no oil resources, but that is sustained by a backstop technology, there exists at least a steady state.

The shape of the curve $G: \rho_0 \rightarrow G(\rho_0), 0 \leq \rho_0 \leq \rho^{\max}(\delta)$ – first rising from 0, then returning to 0 – suggests a possible rich dynamics. Depending on the preferences, the technology, and the values of their parameters, convergence to a steady state might be monotone or in damped oscillation. There might even exist cycles.

3.4. Numerical Example

Suppose that the lifetime utility function is

$$[c^0]^{1-\sigma} / [1-\sigma] + \gamma [c^1]^{1-\sigma} / [1-\sigma] - (b - b^{\max})^2 / (2\beta),$$

and the output produced by one worker – as a function of the energy input – is assumed to be given by $f(e) = ae^\alpha$. For the simulation exercise, the numerical values chosen for the parameters are: $a = 4, \alpha = 0.5, \sigma = 0.5, \beta = 25, \gamma = 0.65, h = 0.25, b^{\max} = 9, \delta = 0.75$.

Also, the initial backstop capital labor ratio is taken to be $\kappa_0 = 0.09$. The following figure depicts the curve $G : \rho_0 \rightarrow G(\rho_0), 0 \leq \rho_0 \leq \rho^{\max}$ for this numerical example.

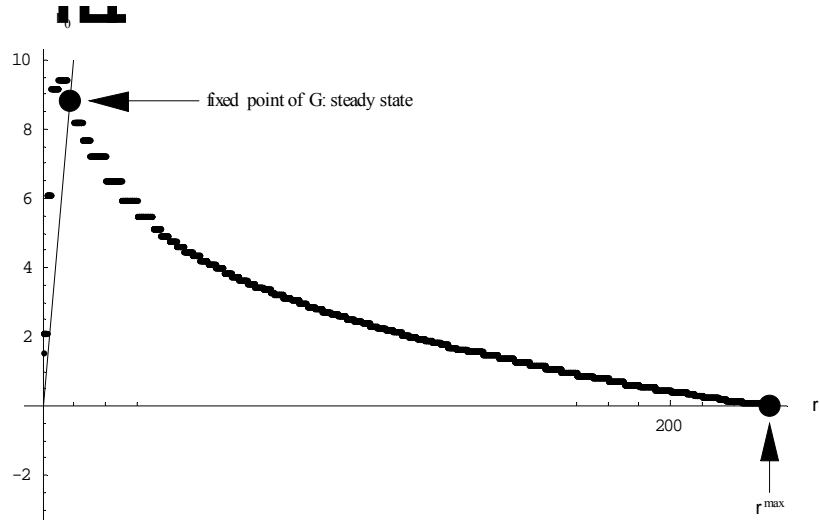


FIGURE.— *The transition of the rental rate of capital from one period to another*

Our calculations show that the critical energy input per worker is $e^{\min} = 0.07$, which yields the following values for the critical wage rate and the critical rental rate of backstop capital: $\omega^{\min} = 0.53$ and $\rho^{\max} = 7.50$. Also, $G^{\max} = 2.89 < \rho^{\max}$, and Assumption 3 is satisfied. The largest confining interval in which the rental rate of backstop capital, i.e., the price of renewable energy, evolves is $[0, G^{\max}] = [0, 2.89]$. The results of the simulation exercise are presented in the following table:

TABLE I

CONVERGENCE TO STEADY STATE IN DAMPED OSCILLATION

($a = 4, \alpha = 0.5, \sigma = 0.5, \beta = 25, \gamma = 0.65, h = 0.25, b^{\max} = 9, \delta = 0.75, \kappa_0 = 0.09$)

Period	κ_t (capital/labor ratio)	ρ_t (rental rate of capital)	ω_t (wage rate)	b_t (birth rate)
0	0.09	6.667	0.6	0.003
1	38.53	0.322	12.41	6.306
2	0.45	2.993	1.34	0.714
3	0.83	2.201	1.82	1.363

4	0.59	2.600	1.54	0.988
5	0.69	2.406	1.66	1.157
6	0.64	2.503	1.60	1.069
7	0.66	2.455	1.63	1.112
8	0.65	2.479	1.61	1.090
9	0.66	2.467	1.62	1.100
10	0.65	2.473	1.62	1.096
11	0.66	2.467	1.62	1.098
12	0.66	2.471	1.62	1.097

The economy converges to a steady state in about 12 periods, and the convergence is in damped oscillation. The capital labor ratio enters a small neighborhood of its steady state value in about 5 periods. The rapid convergence is due to the assumption on the sub-utility function of offspring. When the capital labor ratio is low, the low labor income induces a young individual to give more weight to future consumption at the expense of the number of offspring, resulting in a higher capital/labor ratio in the next period. In the simulation exercise, the initial capital labor ratio has been chosen to be rather low, which induces an initial optimal birth rate of 0.003. The capital labor ratio in period 1 is 38.53, which is high, and the resulting high wage in that period induces the young generation of that period to raise more children. The number of offspring raised by a young individual of period 1 is 6.306, which helps to drive down the capital labor ratio in period 2. The special features of the sub-utility function of offspring thus have a stabilizing influence on the economy, and prevent the population from an abrupt collapse.

In the long run, the population grows at the rate of 9.743% per period. If the cost of raising children is high, then the marginal utility offspring is low, or the productivity of backstop capital is low, the steady state might involve a contracting population, i.e., a steady birth rate strictly less than 1. In this case, the economy will become extinct in the long run. Indeed, if the saturation number of offspring is $b^{\max} = 8$ instead of $b^{\max} = 9$, the steady-state birth rate will be 0.96. A lower value of the saturation number of offspring implies that parents have less love for children, which leads to a birthrate below the replacement rate. If the saturation number of offspring assumes the value of

$b^{\max} = 8.28234$, then the steady-state birthrate is $\bar{b} = 1$, i.e., the population becomes stable in the long run. If the parameters assume the following values:

$$a = 7.45, \alpha = 0.75, \sigma = 0.5, \beta = 15, \gamma = 0.75, h = 0.45, b^{\max} = 10, \delta = 1.0,$$

then the economy has a stable two-cycle $(\rho^*, \rho^{**}) = (4.610, 5.330)$, with the rental rate of backstop capital alternating between 4.610 and 5.330. In terms of birthrates, the two-cycle is $(b^*, b^{**}) = (1.612, 0.623)$, which indicates that the population will grow by 0.4% every two periods.

4. EXISTENCE OF COMPETITIVE EQUILIBRIUM FOR AN ECONOMY WITH OIL RESOURCES

Suppose that the economy begins at time 0 in state (X_0, K_0, N_0^0, N_0^1) , with $X_0 > 0, K_0 \geq 0$. Let $\xi_0 = X_0 / N_0^0$ and $\kappa_0 = K_0 / N_0^0$ denote the initial oil endowment/labor ratio and the initial capital/labor ratio, respectively. To keep the population from becoming extinct in period 1, we shall assume that $\xi_0 + \kappa_0 > e^{\min}(\delta)$.

Let T be a non-negative integer. If we truncate our economy at the end of period T , then we obtain an economy with a finite time horizon that we call the truncated economy with time horizon T . A price system for the truncated economy with time horizon T is a finite sequence $\mathbf{P}^T = (\phi_t, \varphi_t, \omega_t, \rho_t)_{t=0}^T$. An *allocation induced by* \mathbf{P}^T is a list of finite sequences

$$\mathbf{A}^T = (c_0^1, (c_t^0, c_{t+1}^1, b_t, x_{t+1}, k_{t+1})_{t=0}^{T-1}, (Q_t, S_t, L_t, Y_t)_{t=0}^T, (K_t^\#, S_t^\#)_{t=0}^T, (X_t, K_t, N_t^0, N_t^1)_{t=0}^T, c_T^0)$$

with the following properties:

- (i) $c_0^1 = [\phi_0 X_0 + (1 - \delta + \rho_0) K_0] / N_0^1$;
- (ii) $(c_t^0, c_{t+1}^1, b_t, x_{t+1}, k_{t+1})$ is the optimal lifetime plan for a young individual of period t , when the price system \mathbf{P}^T prevails.
- (iii) (Q_t, S_t, L_t, Y_t) is an optimal production plan of the representative firm in the consumption good sector in period t , when the price system \mathbf{P}^T prevails.
- (iv) $(K_t^\#, S_t^\#)$ is an optimal production plan for the representative producer of solar energy in period t , when the price system \mathbf{P}^T prevails.

$$(v) (X_t, K_t, N_t^0, N_t^1) = N_{t-1}^0(x_t, k_t, b_{t-1}, 1), t = 1, 2, \dots, T.$$

$$(vi) c_T^0 = \omega_T.$$

Observe that (vi) represents the consumption of a young individual in period T . Because the problem ends at the end of period T , a young individual of this period has no future to plan for and thus will neither save nor raise children; she will consume all the wages she earns. The pair $(\mathbf{P}^T, \mathbf{A}^T)$ is said to constitute *a competitive equilibrium for the truncated economy with time horizon T* if the following market-clearing conditions are satisfied:

$$(vii) X_{t+1} + Q_t = X_t, 0 < t < T, \text{ and } X_T = Q_T.$$

$$(viii) S_t = S_t^\#, 0 \leq t \leq T,$$

$$(ix) L_t = N_t^0, 0 \leq t \leq T,$$

$$(x) K_t^\# = K_t, 0 \leq t \leq T,$$

$$(xi) [\phi_t X_t + (1 - \delta + \rho_t) K_t] + [N_t^0(c_t^0 + b_t + k_{t+1})] = Y_t + (1 - \delta) K_t, 0 \leq t < T,$$

$$[\phi_T X_T + (1 - \delta + \rho_T) K_T] + [N_T^0 c_T^0] = Y_T + (1 - \delta) K_T.$$

In the technical addendum to this paper, the existence of a competitive equilibrium for a truncated economy and the fact that all the equilibrium prices are positive are proved by using the traditional Debreu-Gale-Nikaido technique. The existence of a competitive equilibrium under infinite time horizon – as asserted by Proposition 3 – is then proved by using Cantor's diagonal trick to show that in the limit, when $T \rightarrow \infty$, a sequence of truncated economies converges, and its limit is a competitive equilibrium under infinite time horizon. In Proposition 3, we let $\xi_t = X_t / N_t^0$, $\kappa_t = K_t / N_t^0$, and $q_t = Q_t / N_t^0$ denote, respectively, the equilibrium oil endowment labor ratio, the equilibrium capital labor ratio, and the equilibrium oil input per worker – all in period $t, t = 0, 1, \dots$

PROPOSITION 3: *Consider an economy with a positive stock of oil and possibly a positive stock of backstop capital. This economy has a competitive equilibrium, say (\mathbf{P}, \mathbf{A}) , with*

$$\mathbf{P} = (\phi_t, \varphi_t, \omega_t, \rho_t)_{t=0}^\infty,$$

$$\mathbf{A} = (c_0^1, (c_t^0, c_{t+1}^1, b_t, x_{t+1}, k_{t+1})_{t=0}^\infty, (Q_t, S_t, L_t, Y_t)_{t=0}^\infty, (K_t^\#, S_t^\#)_{t=0}^\infty, (X_t, K_t, N_t^0, N_t^1)_{t=0}^\infty),$$

with the following properties:

(i) The birthrate in each period is positive.

(ii) The following relationship holds between the price of oil and the price of renewable energy

$$(9) \quad 0 < \min\{\phi_t, \rho_t\} = \rho_t = f'(q_t + \kappa_t) \leq \phi_t < \rho^{\max}, \quad (t = 0, 1, \dots),$$

with equality holding for the second inequality in (9) if $q_t > 0$.

5. OIL EXTRACTION UNDER COMPETITIVE EQUILIBRIUM

To alleviate some of the technical arguments concerning the limiting behavior of the economy when the energy endowments/worker ratio is extremely high or close to the critical level $e^{\min}(\delta)$, we shall make the following assumption

ASSUMPTION 4: For any $e > 0$, we have $f(e) - ef'(e) \leq \overset{\circ}{b}ef'(e)$, where $\overset{\circ}{b}$ is a constant satisfying $1 < \overset{\circ}{b} < b^{\max}$.

Assumption 4 asserts that the earnings of the factor labor relative to the earnings of the factor energy are not very high. The following lemma gives some properties of the competitive equilibrium.

LEMMA 3: (i) If ξ_t is large, but κ_t is not, then q_t is will be large, and a young individual of period t will put all her saving in oil. (ii) If $\xi_t + \kappa_t$ is large, then $q_t + \kappa_t$ is large, and b_t is close to b^{\max} . Furthermore, $\xi_{t+1} + \kappa_{t+1}$ is also large, but $\xi_{t+1} + \kappa_{t+1} < q_t + \kappa_t \leq \xi_t + \kappa_t$.

PROOF: If ξ_t is large, but q_t is not, then the energy input per worker, namely $q_t + \kappa_t$, will be bounded above, which in turn implies that the equilibrium wage rate ω_t is bounded above and the equilibrium price of oil ϕ_t is bounded below. Hence the value of the remaining oil stock per worker at the end of period t , namely $\phi_t(\xi_t - q_t)$, will exceed her labor income, and this situation cannot arise in equilibrium. We have just proved the first part of (i).

To prove (ii), suppose that $\xi_t + \kappa_t$ is large. If κ_t is large, then obviously the energy input per worker $q_t + \kappa_t$ is also large. If κ_t is not large, then ξ_t must be large, and

according to the first part of (i) just proven, q_t must be large. The high energy input per worker in period t leads to a high wage rate in that period, which will induce a young individual of period t to consume more at the present time and raise a number of offspring close to the saturation level. To show that $\xi_{t+1} + \kappa_{t+1}$ is large, suppose that $\xi_{t+1} + \kappa_{t+1}$ remains bounded above when $\xi_t + \kappa_t$ tends to infinity. In this case, the price of energy in period $t + 1$ will be bounded below, and the capital income of an old individual in period $t + 1$ will be bounded above. In such a case, a young individual of period t can certainly increase her lifetime utility by increasing her saving without bound as her labor income rises without bound, contradicting the premise that $\xi_{t+1} + \kappa_{t+1}$ is bounded above.

To show $\xi_{t+1} + \kappa_{t+1} < q_t + \kappa_t \leq \xi_t + \kappa_t$, note that

$$(10) \quad s_t / b_t = \phi_t \xi_{t+1} + \kappa_{t+1} = f'(q_t + \kappa_t) [\phi_t \xi_{t+1} / f'(q_t + \kappa_t) + \kappa_{t+1} / f'(q_t + \kappa_t)] < \omega_t / b_t \\ < \overset{\circ}{b}(q_t + \kappa_t) f'(q_t + \kappa_t) / b_t,$$

where that the last inequality has been obtained by invoking Assumption 4. It follows from the two strict inequalities in (10) and result $\phi_t \geq f'(q_t + \kappa_t)$ that

$$(11) \quad \xi_{t+1} + [\kappa_{t+1} / f'(q_t + \kappa_t)] \leq [\phi_t / f'(q_t + \kappa_t)] \xi_{t+1} + [\kappa_{t+1} / f'(q_t + \kappa_t)] < [\overset{\circ}{b} / b_t](q_t + \kappa_t).$$

Now recall that when $\xi_t + \kappa_t$ is large, $q_t + \kappa_t$ will be large, and b_t will be close to b^{\max} . Furthermore, when $q_t + \kappa_t$ is large, $f'(q_t + \kappa_t)$ will be small, and it follows from (11) that $\xi_{t+1} + \kappa_{t+1} < \xi_{t+1} + [\kappa_{t+1} / f'(q_t + \kappa_t)] < q_t + \kappa_t$, as desired.

Finally, to prove the second part of (i), suppose that ξ_t is large, but κ_t is not. Then q_t is large, according to the first part of (i), and $\xi_{t+1} + \kappa_{t+1}$ is also large, according to the first part of (ii). Furthermore, the price of energy in period $t + 1$ is higher than the price of energy in period t , i.e.,

$$(12) \quad \phi_{t+1} \geq f'(q_{t+1} + \kappa_{t+1}) \geq f'(\xi_{t+1} + \kappa_{t+1}) > f'(q_t + \kappa_t) = \phi_t.$$

If $\kappa_{t+1} > 0$, there are two cases to consider: (i) κ_{t+1} is large when ξ_t is large and (ii) κ_{t+1} remains bounded when ξ_t becomes indefinitely large. In case (i), the rental rate of capital in period $t + 1$, namely $\rho_{t+1} = f'(q_{t+1} + \kappa_{t+1})$, will be close to 0, which implies that the rate of return to capital investment will be close to $1 - \delta \leq 1$. However, according to (12), we

have $\phi_{t+1}/\phi_t > 1$, i.e., for a young individual of period t , the rate of return to oil investment is greater than 1, and it will not be optimal for her to invest in oil. Case (i) thus cannot arise in equilibrium. In case (ii), ξ_{t+1} will be large, which implies that the price of energy in period $t+1$ will be low, and investing in capital will yield a rate of return close to $1 - \delta$, which, again according to (12), is also strictly lower than the rate of return to oil investment.

■

LEMMA 4: *There exist two values, say ρ^- and ρ^+ , which satisfy $0 < \rho^- < \rho^+ < \rho^{\max}(\delta)$, and which do not depend on the rate of capital depreciation, such that $\rho^- < f'(q_t + \kappa_t) < \rho^+$, for all $t = 0, 1, \dots$*

PROOF: Note that when ρ_t is in a small neighborhood of $\rho^{\max}(\delta)$ the equilibrium wage rate ω_t will be in a small right neighborhood of the critical level $\omega^{\min}(\delta)$, and the saving offspring ratio chosen by a young individual of period t will be high, which, according to Lemma 3, leads to a lower value of energy in period $t+1$, i.e., $\rho_{t+1} < \rho_t$. A limiting argument³ can then be used to assert the existence of ρ^+ . Now for $0 < \rho_t < \rho^+$, if ρ_t is small, then $q_t + \kappa_t$ is large, and according to Lemma 3, $q_t + \kappa_t > q_{t+1} + \kappa_{t+1}$, leading to $\rho_{t+1} > \rho_t$. A limiting argument can then be used to assert the existence of ρ^- . ■

From the perspective of a young individual, the decision on whether to invest in oil or capital depends on the rates of return of these assets. For capital investment, a high rate of depreciation discourages capital investment, while a low rate of depreciation, ceteris paribus, makes this asset relatively more attractive than oil. Thus when the rate of depreciation is low, we expect capital investment to be favored over oil investment; the successive generations prefer to invest only in capital, and we can expect that the oil stock will be exhausted in finite time. The following proposition confirms this intuition.

PROPOSITION 4: *If the rate of capital depreciation is not too high, then there exists a competitive equilibrium under which the oil stock is exhausted in finite time.*

³ For more details on the technical arguments, see the technical addendum.

Furthermore, there exists no competitive equilibrium under which the oil stock is exhausted asymptotically.

PROOF: First, we claim that if the rate of capital depreciation is not too high, say $\delta < \rho^-$, then there exists an integer \bar{T} such that for any integer $T > \bar{T}$ and any competitive equilibrium of the truncated economy with time horizon T , the oil stock is exhausted in or before the penultimate period. Indeed, if the claim is not true, then for any positive integer n there exists a positive integer $T, T > n$, and a competitive equilibrium for the truncated economy with time horizon T , say

$$(\mathbf{P}^T, \mathbf{A}^T) = \left(\begin{array}{l} (\phi_t, \varphi_t, \omega_t, \rho_t)_{t=0}^T, \\ (c_0^1, (c_t^0, c_{t+1}^1, b_t, x_{t+1}, k_{t+1})_{t=0}^{T-1}, (Q_t, S_t, L_t, Y_t)_{t=0}^T, (\hat{K}_t, \hat{S}_t)_{t=0}^T, (X_t, K_t, N_t^0, N_t^1)_{t=0}^T, c_T^0 \end{array} \right)$$

such that $X_T > 0$. Because $X_T > 0$, the oil investment of every young generation before the last period must be positive, which implies that the price of oil must rise through time at a rate greater than or equal to the rate of capital investment, i.e.,

$$\phi_{t+1} / \phi_t \geq 1 - \delta + \rho_{t+1}, \quad (t = 0, \dots, T-1).$$

In particular, we have $\phi_T / \phi_{T-1} \geq 1 - \delta + \rho_T$. Because oil exhaustion always occurs in a truncated economy, all of the remaining oil resources at the beginning of period T must be extracted for use in the consumption good sector, and this will constrain the price of oil in period T not to exceed the rental rate of capital in that period, i.e., $\phi_T \leq \rho_T$. Using this last result in the preceding inequality, we obtain $\phi_T / \phi_{T-1} (1 - \phi_{T-1}) \geq 1 - \delta$, which constrains the price of oil in the penultimate period to be bounded above by 1, i.e., $\phi_{T-1} < 1$. However, we know that the price of oil must rise through time from the initial level $\phi_0 = f'(q_0 + \kappa_0) \geq f'(\xi_0 + \kappa_0) > 0$ at or above the rate of return to capital investment. Furthermore, according to Lemma 4, we must have $\rho_t > \rho^-$, for $t = 0, \dots, T$. Hence, using the hypothesis $\delta < \rho^-$, we obtain $1 - \delta + \rho_t > 1 - \delta + \rho^- > 1$, for $t = 0, \dots, T$. This last result implies that the price of oil in period $T-1$ will be arbitrarily large when T is large, which contradicts the hypothesis of the reductio ad absurdum argument. The claim is now established.

We are now ready to prove Proposition 4. To this end, note that in the sequence of truncated economies used in the proof of Proposition 4, oil resources are depleted by

period \bar{T} in all the truncated economies with time horizon greater than or equal to \bar{T} . Hence in the economy that is the limit of a subsequence of the sequence of truncated economies oil exhaustion also occurs by period \bar{T} . This proves part (i) of Proposition 4. To prove (ii) of Proposition 4; note that if the oil stock is exhausted asymptotically, then the price of oil must rise indefinitely through time at or above the rate of return to capital investment. The price of oil thus will tend to infinity when t tends to infinity. However, we have already argued in the proof of the claim that the price of oil in the period preceding a period in which oil is extracted for use in the production of the consumption good is bounded above by 1. Thus, part (ii) of Proposition 4 is established. ■

6. INCOMPLETE OIL EXHAUSTION UNDER COMPETITIVE EQUILIBRIUM

In the preceding section, we show that when the rate of capital depreciation is not too high, there exists a competitive equilibrium under which oil exhaustion occurs in finite time. The following question immediately arises. Are there equilibria under which part of the oil stock is left in situ unexploited. The answer to this question is negative is maybe, as illustrated by the numerical example given in Sub-section 6.2 below.

6.1. Steady States under Incomplete Oil Exhaustion

Consider a competitive equilibrium under which the oil stock is partially depleted, and that T is the last period the oil stock is exploited. Then we have $0 < Q_T < X_T$ and $Q_t = 0, t > T$. Because all the young generations of period T and after put their savings in both oil and capital, we must have

$$(13) \quad \phi_{t+1} / \phi_t = 1 - \delta + \rho_{t+1}, \quad (t \geq T).$$

Furthermore, the current budget constraint for a young individual of period $t \geq T$ can be expressed under the following form:

$$(14) \quad \phi_t X_T / N_t^0 = \omega_t - c_t^0 - hb_t - k_{t+1}.$$

Forwarding (14) by one period, then dividing the result by (14), we obtain

$$(15) \quad \frac{\phi_{t+1} N_t^0}{\phi_t N_{t+1}^0} = \frac{1 - \delta + \rho_{t+1}}{b_t} = \frac{\omega_{t+1} - c_{t+1}^0 - hb_{t+1} - \kappa_{t+2} b_{t+2}}{\omega_t - c_t^0 - hb_t - \kappa_{t+1} b_{t+1}}.$$

Note that in (15) we have let $\kappa_t = k_t / b_t$ denote the capital/labor ratio in period $t, t \geq T$, and have used (14) to obtain the first equality. Also, note that the second equality in (15) is a second-order nonlinear difference equation in the capital/labor ratio $\kappa_t, t \geq T$. If

$$(16) \quad \bar{k} = \lim_{t \rightarrow +\infty} \kappa_t$$

exists, then in the limit, the second equality in (15) becomes

$$(17) \quad [1 - \delta + \bar{\rho}] / \bar{b} = 1,$$

where we have let $\bar{\rho} = \lim_{t \rightarrow +\infty} \rho_t$ and $\bar{b} = \lim_{t \rightarrow +\infty} b_t$.

If the rate of capital depreciation is not too high, then $\bar{\rho} > \delta$ according to Lemma 4, and we must have $\bar{b} = 1 - \delta + \bar{\rho} > 1$, which means that in steady state the population and the price of oil all grow at a rate equal to the rate of return to capital investment. Furthermore, a young individual of any period owns only a fraction of the oil owned by her parent, with the fraction being the inverse of the number of children raised by the parent: the same oil stock is owned by each of the successive young generations, and due to population growth each young individual in later periods owns a smaller and smaller part of the economy's oil stock.

When is incomplete oil exhaustion a likely outcome under competitive equilibrium? To answer this question, let us look at the following more detailed representation of the division of output among the various uses in a steady state under incomplete oil exhaustion:

$$(18) \quad (\bar{s} - \bar{b} \bar{\kappa}) = f(\bar{k}) - \bar{\kappa} f'(\bar{\kappa}) - \bar{c}^0 - h \bar{b} - \bar{b} \bar{\kappa}.$$

In (18), we have let \bar{s} represent the saving of a young individual. The left side of (18) thus represents the funds allocated to oil investment. The right side of (18) represents what remains of the output of the consumption good produced per worker after (i) the factor capital has received its remuneration and (ii) the young individual has paid for her current consumption; the costs of raising children, and the cost of capital investment required to sustain the steady state of the economy. If the earning of capital relative to output is high, there will be little left for wages. Furthermore, out of the low wages, the young individual must pay for her current consumption, the cost of raising children, and capital investment. Because the birthrate is higher than 1, the cost of raising children will be substantial if the cost of raising a child is high. There might not exist any value of $\bar{\kappa}$

such that the right side of (18) is positive, a necessary condition for incomplete oil exhaustion. When such a value exists, one can always construct a competitive equilibrium under which the oil stock is only partially exploited. Proposition 5 stated below gives a condition for the oil stock to be partially exploited. In this proposition, $\rho(\kappa) = f'(\kappa)$ and $\omega(\kappa) = f(\kappa) - \kappa f'(\kappa)$ denote, respectively, the rental rate of capital and the wage rate that prevail when only renewable energy is used in the production of the consumption good and when κ is the capital labor ratio. Also, recall that $c^0(\omega_t, r_{t+1})$ and $b(\omega_t, r_{t+1})$ denote, respectively, the current consumption of a young individual of period t and the number of children she raises, given that ω_t is the prevailing wage rate in period t and r_{t+1} is the rate of return to her saving.

PROPOSITION 5: *If the inequality*

$$(19) \quad f(\kappa) - \kappa\rho(\kappa) - c^0(\omega(\kappa), 1 - \delta + \rho(\kappa)) - (h + b(\omega(\kappa), 1 - \delta + \rho(\kappa)))\kappa > 0$$

is satisfied for some value of $\kappa > e^{\min}(\delta)$, then there exist infinitely many steady states in which part of the oil stock is left in situ unexploited. Furthermore, the capital/labor ratio and the birthrate are lower in a steady state with incomplete oil exhaustion than in the steady state with complete oil exhaustion.

PROOF: Let κ_0 be a value of κ that satisfies (19) and

$$(20) \quad \xi_0 = \frac{1}{\phi_0} \left[f(\kappa_0) - \kappa_0\rho(\kappa_0) - c^0(\omega(\kappa_0), 1 - \delta + \rho(\kappa_0)) - (h + b(\omega(\kappa_0), 1 - \delta + \rho(\kappa_0)))\kappa_0 \right]$$

where we have let $\phi_0 = \rho(\kappa_0)$. Here we shall interpret κ_0 as the initial capital labor ratio and ξ_0 – defined by (20) – as the initial oil endowment per worker. Next, let $b_0 = b(\omega(\kappa_0), 1 - \delta + \rho(\kappa_0))$, and suppose that in period $t, t = 0, 1, \dots$, the price of oil, the rental rate of capital, and the wage rate are given, respectively, by $\phi_t = \phi_0 b_0^t$, $\rho_t = \rho(\kappa_0)$, $\omega_t = \omega(\kappa_0)$. It is straightforward to verify that when the price system $\mathbf{P} = (\phi_t, \rho_t, \omega_t)_{t=0}^\infty$ prevails, a young individual of each period $t = 0, 1, \dots$, will have the same labor income, will have the same current and old-age consumption, will raise the same number of children, will invest in the same quantity of capital per child, and will spend the same amount of real resources to buy oil. The price system thus constructed and the lifetime

plans induced by this price system thus constitute a competitive equilibrium. Under this competitive equilibrium, the oil stock is never exploited; the capital/labor ratio is constant; the birthrate is constant; the rate of return to capital is equal to the birthrate; and the price of oil rises through time geometrically at a rate equal to the birthrate. The competitive equilibrium thus constructed is thus a steady state for an economy with exhaustible resources.

Now note that if there exists a value of κ that satisfies (19), then by continuity all the capital/labor ratios in a small neighborhood of κ also satisfy (19), which implies that if there exists one steady state, then there exist infinitely many steady states. Finally, note that when κ is high, the left side of (19) will be negative due to the Inada condition $\lim_{\kappa \rightarrow +\infty} f'(\kappa) = 0$. Let $\underline{\kappa}$ be the smallest value of κ such that the left side of (19) is less than or equal to 0. Then any value of κ that satisfies (19) will be strictly less than $\underline{\kappa}$; that is, the capital labor ratio in a steady state with incomplete oil exhaustion is less than that in the steady state with complete oil exhaustion. The lower capital labor ratio in a steady state with incomplete oil exhaustion means a lower wage rate and a higher rate of return to saving. These last results imply – according to Assumption 2 – a lower birthrate in a steady state with incomplete oil exhaustion than in the steady state with complete oil exhaustion. ■

Incomplete oil exhaustion is likely to exist if wages account for a proportion that is much higher than capital remuneration and if the cost of raising a child is low. If oil resources are abundant, the oil input per worker will be high according to Lemma 3. Thus the amount of oil left in situ unexploited in the case of incomplete oil exhaustion will be relatively small so that in equilibrium successive young generations can afford to pay for the investment in this asset out of their wages. A competitive equilibrium with incomplete oil exhaustion is obviously not Pareto efficient. Although oil has an intrinsic value as an input in the production of the consumption good, the part of the oil stock left unexploited serves no production purposes. Its only use is a store of value, a means through which successive young generations transfer their incomes made during their working days to days of retirement. In this manner, the part of the oil stock left unexploited serves a basic function of money: a store of value. In contrast with paper

money, which has no intrinsic value and might have a zero price in equilibrium,⁴ oil left under the ground unexploited always has a positive value, which, according to our version of Hotelling rule in general equilibrium setting, must appreciate at the rate of interest, namely the real value of the solar energy harnessed from the marginal unit of capital. This surprising feature – an oil bubble so to speak – is first encountered here.

6.2. Numerical Example

Suppose that preferences are represented by the following lifetime utility function: $\text{Log}c^0 + \gamma\text{Log}c^1 + v(b)$, with $v(b) = \beta\text{Log}(b+1)$, $0 \leq b \leq \hat{b}$, where β is a positive parameter and \hat{b} is a constant greater than 1 but less than the saturation number of offspring b^{\max} . As specified, the single-period sub-utility function is logarithmic and the sub-utility function of offspring is also logarithmic in the relevant range $[0, \hat{b}]$. The number of offspring, and saving depend only on labor income, not on the rate of return to saving, and the optimal lifetime plan for a young individual of period t is given by

$$c_t^0 = \frac{h + \omega_t}{1 + \beta + \gamma}, \quad c_{t+1}^1 = \frac{\gamma(h + \omega_t)}{1 + \beta + \gamma} r_{t+1}, \quad b_t = \frac{\beta\omega_t - h(1 + \gamma)}{h(1 + \beta + \gamma)}, \quad s_t = \frac{\gamma(h + \omega_t)}{1 + \beta + \gamma}.$$

As for the output of the consumption good produced by a worker, we assume that it is given by $f(e) = e^\alpha$, $0 < \alpha < 1$. The following values for the parameters are assumed: $\alpha = 0.10$, $\beta = 0.73$, $\gamma = 0.77$, $h = 0.13$, and $\delta = 0.15$. Also, the initial oil endowment per worker and the initial capital/labor ratio are assumed to be given by $\xi_0 = 2.011$ and $\kappa_0 = 0$, respectively. For this numerical example, we are able to find two competitive equilibria, one under which the oil stock is exhausted in finite time, and one under which the oil stock is only partially depleted.

Table II presents the equilibrium under which the oil stock is depleted in finite time. Under this equilibrium, the oil stock is completely exhausted at the end of period 5. After that the economy is completely sustained by the backstop. The two technologies co-exist during four periods, and the economy enters a steady state – without any oil left – in

⁴ See McCandless and Wallace (1991, Chapter 10).

period 6. The steady state capital labor ratio is 0.263, and the steady state birthrate is 1.068.

TABLE II
 COMPETITIVE EQUILIBRIUM WITH COMPLETE OIL EXHAUSTION
 ($\alpha = 0.10, \beta = 0.73, \gamma = 0.77, h = 0.13, \delta = 0.15, \xi_0 = 2.011, \kappa_0 = 0$)

Period	ξ_t	κ_t	q_t	ϕ_t	ρ_t	ω_t	b_t
0	2.011	0	0.480	0.194	0.194	0.836	1.179
1	1.418	0	0.453	0.204	0.204	0.831	1.168
2	0.826	0.084	0.337	0.218	0.218	0.825	1.155
3	0.424	0.161	0.222	0.237	0.237	0.818	1.138
4	0.177	0.214	0.127	0.264	0.264	0.808	1.116
5	0.044	0.246	0.044	0.305	0.305	0.795	1.087
6	0	0.261	0	0.335	0.335	0.787	1.068
7	0	0.263	0	0.332	0.332	0.788	1.068
8	0	0.263	0	0.332	0.332	0.788	1.068
...
t	0	0.263	0	0.332	0.332	0.788	1.068

The second equilibrium we find is presented in Table III.

TABLE III
 COMPETITIVE EQUILIBRIUM WITH INCOMPLETE OIL EXHAUSTION
 ($\alpha = 0.10, \beta = 0.73, \gamma = 0.77, h = 0.13, \delta = 0.15, \xi_0 = 2.011, \kappa_0 = 0$)

Period	ξ_t	κ_t	q_t	ϕ_t	ρ_t	ω_t	b_t
0	2.011	0	0.480	0.194	0.194	0.836	1.179
1	1.898	0	0.453	0.204	0.204	0.831	1.168
2	1.253	0	0.421	0.218	0.218	0.825	1.155

3	0.721	0.097	0.287	0.237	0.237	0.818	1.138
4	0.381	0.165	0.175	0.264	0.264	0.808	1.116
5	0.185	0.209	0.081	0.305	0.305	0.795	1.087
6	0.095	0.232	0	0.372	0.372	0.778	1.047
...
t	$\frac{0.095}{b_6^{t-6}}$	0.232	0	$0.372b_6^{t-6}$	0.372	0.778	1.047

As can be seen from Table III, for the first three periods, all the energy requirements of the economy are met by drawing down the oil resources. Capital begins to be accumulated at the end of the third period, and energy produced by the backstop technology provides part of the energy inputs in the fourth period. The two technologies – fossil fuels and the backstop – are both exploited during three periods, with the backstop gradually replacing oil. From period 6 on, the oil stock is not exploited anymore. The amount of oil that remains at the beginning of period 6 is left in situ, unexploited, and all the energy requirements of the economy are met by the backstop technology. The economy enters a steady state at the beginning of period 6. In steady, the capital labor ratio is 0.232, and the population as well as the aggregate capital stock grows geometrically at the rate of 1.047, which is also the rate of return to capital investment. Note that the rate of capital depreciation has been deliberately chosen to be low, $\delta = 0.15$, so that an equilibrium under which the oil stock is depleted in finite time exists, as asserted by Proposition 4. Also, note that the capital share in national income ($\alpha = 0.10$) and the cost of raising a child ($h = 0.13$) are chosen sufficiently low to ensure that an equilibrium with incomplete oil exhaustion exists, as asserted by Proposition 5. Finally, observe that under both equilibria, the convergence to steady state is monotone, due to logarithmic preferences, and that in steady state, the capital labor ratio as well as the birthrate are both lower under the equilibrium with incomplete exhaustion.

7. CHARACTERIZATION OF COMPETITIVE EQUILIBRIUM

To concentrate on the influence of exhaustible resources on fertility decisions and on the process of technology substitution, we shall consider the case in which fossil fuels are abundant, but backstop capital is not. Also, we shall assume that the population is either stable or growing in the long run.

When fossil fuels are abundant at the beginning, a competitive equilibrium consists of three phases. In the first phase, the energy inputs used in the production of the consumption good come solely from oil. Because the population is stable or growing in the long run, oil alone cannot sustain the economy indefinitely, and the backstop must be brought into use at some time to provide part of the energy requirements of the economy. The time interval that encompasses the introduction of the backstop and the end of extraction activities constitutes the second phase of a competitive equilibrium: the phase of technology substitution. The third phase of a competitive equilibrium begins after all extraction activities have been terminated, either due to oil exhaustion or because the competitive equilibrium in question involves incomplete oil depletion.

According to Lemma 3, the oil input per worker in period 0 will be high, which means a high wage rate in this period. The high labor income will induce a young individual of period 0 to raise a number of children close to the saturation level b^{\max} . The division of the output of the consumption good produced by a worker between the two factors of production is represented by the identity $f(q_0) = \omega_0 + \phi_0 q_0$. Furthermore, using Assumption 4 and the fact that $b_0 > \overset{\circ}{b}$, we can assert that $\omega_0 < b_0 \phi_0 q_0$. Also, according to Lemma 3, a young individual of period 0 will invest all her saving in oil, and we have

$$(21) \quad \xi_1 = [\xi_0 - q_0] / b_0 = [s_0 / b_0 \phi_0] < [\omega_0 / b_0 \phi_0] < q_0 < \xi_0.$$

The chain of inequalities in (21) indicate that the oil endowment per worker in period 1 is strictly less than the oil input per worker in period 0, which is in turn less than the oil endowment per worker in period 0. Because $q_1 \leq \xi_1$, we must have

$$(22) \quad \phi_1 = f'(q_1) \geq f'(\xi_1) > f'(q_0) = \phi_0,$$

i.e., the price of oil in period 1 will be higher than the price of oil in period 0. If the oil endowment per worker in period 1 is still large, the preceding argument can be repeated

to assert that the oil input per worker in period 1 – although lower than that in period 0 – is still high, and a version of (21) as well as a version of (22) also hold for period 2.

During the early phase of the competitive equilibrium, the oil endowment per worker falls rapidly. There are two reasons behind this fast decline. First, the price of oil must be low to clear the oil market. More precisely, the low price of oil induces the firms producing the consumption good to use more of this input. The high oil input per worker also means a high wage rate, allowing the young generation to save more, which, coupled with a low oil price, make it possible for the young generation to buy the rest of the oil stock as investment. Second, the initial high birthrates mean less oil is available for each worker in the following periods. One implication of the fast decline in the oil endowment per worker is a slow-down in the population growth. As the price of oil rises, the wage rate declines, and this in turn induces a fall in the birthrate. If the rate of return to oil investment rises through time, the substitution effect – according to Assumption 2 – will reinforce the income effect and cause the birthrate to fall even further. Thus we can expect the birthrate – which is high at the beginning – to decline steadily through time as the oil stock is being exploited.

In the second phase, technology substitution – backstop for fossil fuels – takes place. When both technologies are exploited in a period, say t , we must have

$$(23) \quad \phi_t / \phi_{t-1} = 1 - \delta + \rho_t = 1 - \delta + \phi_t.$$

Observe that when (23) holds, we must have $\phi_{t-1} < 1$. Furthermore, according to Lemma 4, if the rate of capital depreciation is not too high, then the right side of the second equality in (23) will be greater than 1, which implies that the price of oil as well as the rate of return to oil and capital both rise during the phase of technology substitution. Again as our discussion of the first phase, the rise in energy prices means a fall in the wage rate. Furthermore, a rise in the return to saving will induce a young individual to save more at the expense of children, according to Assumption 2. Thus, the birthrate continues to decline in the second phase.

How long does the second phase last? To answer this question, one must determine precisely the time ϕ_{t-1} exceeds 1, which requires many more technical arguments that we have not carried out. Needless to say, the length of the technology substitution phase

depends critically on the rate of capital depreciation. In the particular case of $\delta = 1$, (23) is reduced to $\phi_t / \phi_{t-1} = \phi_t$, which leads to $\phi_t = \phi_{t-1} = 1$; that is when capital depreciates completely at the end of each period, a necessary condition for the two technologies to co-exist for a period is that the price of oil in that and in the previous period to be equal to 1, a result that cannot possibly arise in equilibrium. Thus, when capital depreciates completely, technology substitution occurs abruptly, with the backstop being brought into use only after the fossil fuels have been exhausted. The second phase does not exist in this case. This result is not hard to understand. When capital depreciates completely, it is not different from oil – an exhaustible resource – from the perspective of an investor: both fetch the same price on the energy markets and both are used up at the end of the production process.

During the third phase of a competitive equilibrium all the energy needs of the economy are met by the backstop. There are two possible scenarios to consider: complete oil exhaustion and incomplete oil exhaustion.

If the oil resources have been completely depleted when the third phase begins, then the evolution of the economy from this time on is completely determined by the backstop technology and the preferences, as described in Section 3. Depending on the values of the parameters, the economy might converge to a steady state in a monotone manner, in damped oscillation, or it might converge to a stable cycle. The existence of an exhaustible resource has only a fleeting impact in the short run, especially at the beginning when the resource makes it possible for the population to grow rapidly and for capital to accumulate in a less painful manner. One can visualize through time the process of transforming oil into the consumption good and into backstop capital. In the long run oil does not influence the birthrate; its only impact is to allow for a population with a larger absolute size.

In the case the competitive equilibrium involves incomplete oil exhaustion, there are infinitely many possible steady states that arise from the infinitely many possible equilibria with incomplete oil exhaustion. Starting from a given steady state, one can go back in time to the initial state of the economy. As indicated by Proposition 4, when the rate of capital depreciation is low, there exists also a competitive equilibrium under which the oil stock is exhausted in finite time. The possibility of multiple equilibria arises

from the indeterminate mix of solar energy and oil use that we have pointed out in solving the profit maximization problem represented by in Sub-section 2.2.4. Starting from the same initial oil stock and the same initial capital stock, the economy might evolve along different equilibrium trajectories, reaching the point at which solar energy is substituted for oil at different times. Furthermore, at the time technology substitution takes place, the oil stock and the capital stock under complete oil exhaustion might assume values that are different from those under incomplete oil exhaustion (including the special case the oil stock is completely exhausted when the backstop is first brought into use). Which steady state the economy will converge to in the long run depends on the state of the system at the time the backstop completely replaces oil in the production of the consumption good. In each of these steady states, the capital labor ratio and the birthrate are both lower than those in the state with complete oil exhaustion.

8. CONCLUSION

The major findings of the paper can be summarized as follows. When the economy begins with a positive stock of reproducible capital, but has no oil, there exist a unique competitive equilibrium and a steady state (see our Proposition 1 and 2). Depending on the specified functional form of the lifetime utility function, convergence to the steady state might be monotone or in damped oscillation. Furthermore, by varying the values of some parameters of the model, we might obtain cycles instead of convergence in damped oscillation. We would like to mention in passing that the problem raised by Galor and Ryder (1989) about the non-existence of a steady state is due to the assumption that the population grows at an exogenously given rate. The introduction of fertility decisions into the overlapping-generations model (as can be seen in the proof of Proposition 2) is sufficient to guarantee the existence of a steady state in our model.

The results of the model change dramatically when the economy is endowed with a stock of fossil fuels. In traditional models of resource extraction – à la Hotelling or in the tradition of optimal growth – the resource is always presumed to be completely depleted, and the equilibrium to be efficient; the only concern is to see how the resource is exploited through time. In an overlapping-generations model, this presumption turns out

to be unfounded. When the *time horizon is infinite* and agents make plans only for the two periods of their life cycles, we cannot take for granted that the resource stock will be completely depleted or that the equilibrium will be efficient. In her lifetime plan, an agent does not consider the impact of her decision on the welfare of future generations and can always invest either in backstop capital or in oil, and then sells what she owns to the young generation or the competitive firms of the next period. The fact that the resource is exhaustible has no bearing on her investment decision, and as long as the price of the resource rises through time at the market rate of interest, she is quite willing to hold this asset so that the resource stock might only be partially depleted. The part of the resource stock left in situ unexploited serves as a *financial bubble*, and in steady state the resource price – according our modified version of the Hotelling rule – will rise geometrically through time at a rate equal to the marginal product of capital, which is also the steady-state birthrate.

Incomplete depletion of the resource stock requires two conditions. First, the price of oil must be above the price of renewable energy to discourage its use by the competitive firms. Second, because the value of the stock left in situ must rise through time, the population must also be growing so that the oil investment made by a young individual of each period is only a fraction of the oil owned by a young individual in the preceding period, with the fraction being equal to the inverse of the birthrate. We demonstrate that if the rate of capital depreciation is low, there exists a competitive equilibrium under which the stock of fossil fuels is completely depleted in finite time (see our Proposition 4). We also provide conditions for which there exists a continuum of steady states in which the oil stock is only partially depleted (see our Proposition 5). In our model, the indeterminacy of the steady state arises from the endogenous fertility decision of a young individual and from the indeterminate mix of oil and capital in her investment portfolio.

To see why the indeterminacy of the steady state cannot occur under the assumption that the population grows at an exogenously given rate, first note that according to Hotelling rule, the price of oil must grow geometrically at the rate of interest, which is the marginal product of capital, when oil exhaustion is not complete. Second, note that the amount of oil bought by a young individual in any period is only a fraction of the amount of oil bought by a young individual of the previous period, with the fraction being equal

to the inverse of the exogenous birthrate. Third, in steady state the value of the oil investment of a young individual is constant, and given that the price of oil must rise geometrically at the rate of interest, the amount of oil bought by a young individual of successive generations must decline geometrically at the same rate, which we have shown to be equal to the inverse of the birthrate. Therefore, the steady-state capital labor ratio is completely determined by the exogenous birthrate, and thus indeterminacy of the steady state cannot arise if the population grows at an exogenously given rate.

With endogenous fertility, on the other hand, when the economy is endowed with a stock of fossil fuels, there are two variables to choose from a single budget constraint in steady state: the capital labor ratio and the birth rate. Because the rates of return to oil and capital investments are the same, the investment mix of a young individual is indeterminate, and this means there is freedom in choosing either the level of capital investment or the value of the oil investment. Choosing one means implicitly choosing the other, and this is the reason why the set of possible steady states is a one-dimensional manifold.

In general, we can expect that convergence to different steady states involves starting from different initial conditions. The indeterminacy of the steady state thus provides a possible explanation for the variations in incomes and birthrates across countries along their paths of economic development by appealing to their different initial conditions ⁵. For an economy that converges to a steady state with incomplete depletion, there exists also a competitive equilibrium under which the oil stock is depleted in finite time if the rate of capital depreciation is low. That is, for the same initial condition, there might exist two forward-looking competitive equilibria – one under which the oil stock is depleted in finite time, and one with incomplete depletion. In this case, cultural or non-economic factors can function as an equilibrium selection mechanism. In the long run, the economy that uses part of the oil stock as a store of value to transmit wealth from one generation to the next has a lower birthrate and a lower capital labor ratio than the economy that

⁵ In the traditional Solow growth model, there exists a unique steady state, and convergence to the steady state is global. The model is thus unable to explain the variations in incomes across countries in the long run. A possible explanation for these variations was provided by Lucas (1993), who suggested that the variations in growth rates across countries are due to their different levels of human capital, and human capital is not mobile. Recently, multiple equilibria have emerged as a possible explanation for these variations. See, for example, Yip and Zhang (1997), who tried to generate indeterminacy in a simple endogenous growth model with endogenous fertility to explain these variations.

exhausts its resource stock, presumably because investing in oil leaves fewer resources for raising children and for the capital investment needed to sustain a steady state.

The evolution of an economy endowed with a large stock of fossil fuels can be described as follows. For a given initial stock of capital, a large initial oil stock means a large oil endowment per worker in period 0, which leads to a high oil input (see Lemma 1.(i)) per worker and a fortiori high wages in this period. The high wages induce a young individual of period 0 to increase her current consumption, to raise a more offspring, and to save more for her old-age consumption. Furthermore, all her savings will be put into oil (see Lemma 1.(ii)) because the abundance of fossil fuels discourages the accumulation of capital in the backstop sector. If the oil endowment per worker in period 1, which is lower than that in period 0 because of high fertility in period 0, is still high (see Lemma 1.(iii)), then what happens in period 0 repeats itself in period 1. As long as this process unfolds, the price of oil will be rising; the oil endowment per worker will be declining; and the birthrate will be falling.

If the population is stable or rising in the long run, then oil alone cannot sustain the economy forever, and the backstop must be brought into use at some point in time. The phase that encompasses the period the backstop is first brought into use and the period extraction activities last take place is the transition phase during which the two technologies – fossil fuels and backstop – co-exist, with the backstop progressively displacing fossil fuels in satisfying the energy needs of the economy. During this phase, the price of oil continues to rise, while the birthrate continues to fall.

At the end of the phase of technology substitution, the backstop completely takes over the function of meeting the energy needs of the economy. If oil exhaustion takes place at the end of the transition phase, then in the long run, the capital labor ratio and the birthrate are the same as if the economy were never endowed with a stock of fossil fuels. If the oil stock is only partially depleted, then in steady state the price of oil must rise geometrically at the steady-state birthrate. In the terminology of Tirole (1985), oil *in situ* becomes a financial bubble despite its potential in contributing to real production. The oil stock left in situ serves one function of money – a store of value – and represents another dimension of inefficiency in overlapping-generations models first encountered here.

Our model can be extended to include human capital. In the extended model, the cost of raising children and the sub-utility function of offspring will depend on the number and quality of offspring. The effective labor supply in the next period can then be considered as a composite good that captures both the number of offspring and their quality. Furthermore, for a young individual, the number of offspring is an inferior good, while quality of a child is a superior good, so that the quantity-quality trade-off should favor quantity over quality at low labor income, but quality over quantity at high labor income. For such an extended model, the discovery of a large stock of an exhaustible resource leads to rise in the effective wage rate, which can set in motion the demographic transition – according to Lucas’ the theory of economic development (see Lucas (1998)) – by making the quantity-quality trade-off easier.

Although it is not our aim, the model we formulate sheds some light on policies. The somehow obvious result is the following. In the numerical example of Sub-section 3.2, we show that a shock that induces a decline in the saturation number of offspring from $b^{\max} = 9$ to $b^{\max} = 8$, with the values of the other parameters remaining the same, will result in a steady-state birthrate of 0.96, and the population will becoming extinct in the long run. Our model thus suggests a possible remedy for this problem of population extinction: subsidize the raising of children, and act conversely, in face of population explosion.

This paper does not deal with normative issues which are of importance. It is well known that inefficiency of various kinds usually arise in overlapping-generations models. In addition to the eventual ‘‘under-accumulation’’ of capital, we now face the possibility of oil bubble, an ‘‘over-accumulation’’ of a resource asset. Besides the prescription of affecting the pattern of capital accumulation through taxation imposed on bequests, gifts, etc. in order to restore economic efficiency, the obvious policy implication of our work is how to induce complete resource exhaustion by some public intervention which should, at some point in time, discourage wasteful resource hoarding under the form of *in situ* as a resource bubble. Also, public intervention may alter the resource ownership over time: instead of ‘‘grand fathering’’ the entire resource stock to the fist generation. An authority argues that it is a common property, henceforth distribute the right to use this stock to different future generations; see Gerlag and Keyser (2001), or Valente (2006) for

analyses in the absence of population consideration. Besides efficiency, there is also the inter-generational fairness issue. Restricting to selfish agents with finite lifetime, market evaluation and, therefore, resource depletion are profitable to the current generation, and this is not necessarily compatible with the well-being of future generations; see, for instance, Mourmouras (1993). If some light could be shed with their simple OLG setting, the analytical task is rather involved with our model where population is endogenous. Because of the general equilibrium repercussion of any policy on the rest of the economy, this is rather cumbersome, and would lead us too far afield.

REFERENCES

- AGNANI, B., M. J. GUTIÉRREZ, AND A. IZA (2005): "Growth in Overlapping-Generation Economies with Non-Renewable Resources," *Journal of Environmental Economics and Management* 50, 387-407.
- ALTONJI, J. G., F. HAYASHI, AND I. KOTLIKOFF (1992): "Is the Extended Family Altruistically Linked? Direct Tests Using Micro Data," *American Economic Review*, 82, 1177-1198.
- AZARIADIS, C. (2000): *Intertemporal Macroeconomics*, Blackwell Publishers, UK.
- BARRO, R. J. AND G. S. BECKER (1989): "Fertility Choice in a Model of Economic Growth," *Econometrica* 57, 481-501.
- BECKER, G. S. AND R. J. BARRO (1988): "A Reformulation of the Economic Theory of Fertility," *Quarterly Journal of Economics* 103, 1-25.
- BECKER, G. S., K. MURPHY AND R. TAMURA (1990): "Human Capital, Fertility, and Economic Growth," *Journal of Political Economy*, S12-S37.
- CASS, D. AND T. MITRA (1991): "Indefinitely Sustained Consumption Despite Exhaustible Natural Resources," *Economic Theory* 1, 119-146.
- DASGUPTA, P. AND G. HEAL (1974): "The Optimal Depletion of Exhaustible Resources," *Review of Economic Studies Symposium*, 3-28.
- _____ (1979): *Economic Theory and Exhaustible Resource*, Cambridge University Press.
- ECKSTEIN, Z., S. STERN, AND K. WOLPIN (1988): "Fertility Choice, Land and the Malthusian Hypothesis," *International Economics Review* 29, 353-361.

ERHLICH, I. AND F. T. LUI (1991): "Intergenerational Trade, Longevity, and Economic Growth," *Journal of Political Economy* 98, 761-782.

_____ (1997): "The Problem of Population and Growth: A Review of the Literature from Malthus to Contemporary Models of Endogenous Population and Endogenous Growth," *Journal of Economic Dynamics and Control* 21, 205-242.

GALOR, O. AND H. E. RYDER (1989): "Existence, Uniqueness, and Stability in Overlapping-Generation Model with Productive Capital," *Journal of Economic Theory*, 49, 360-375.

GALOR, O. AND D. WEIL (2000): "Population, Technology, and Growth: From the Malthusian Regime to the Demographic Transition and Beyond," *American Economic Review*, 90, 806-828.

_____ (2000): "From Malthusian Stagnation to Modern Growth," *American Economic Review*, 89, 150-154.

GENEAKOULOPOLOS, J. D. AND H. M. POLEMARCHAKIS (1991): "Overlapping Generations, in W. Hildrebrand and H. Sonenschen (Eds.) *Handbook of Mathematical Economics IV*, North Holland, Amsterdam, chapter 35.

GERLAG, R. AND M. A. KEYZER (2001): "Sustainability and the Intergenerational Distribution of Natural Resource Entitlements," *Journal of Public Economics*, 79, 315-341.

HUNG, N. M. AND N. V. QUYEN (1993): "On R & D Timing Under Uncertainty," *Journal of Economic Dynamics and Control* 17, 971-991.

_____ (1994): *Dynamic Timing Decisions Under Uncertainty*, Springer Verlag.

KEMP, M. AND N. V. LONG (1979): "The Under-Exploitation of Natural Resource: A Model with Overlapping Generations," *Economic Record* 55, 214-221.

KRAUTKRAEMER, J. M (1998): "Nonrenewable Resource Scarcity," *Journal of Economic Literature*, 36, 2065-2107.

KRUTILLA, K. AND R. REUVENY (2006): "The Systems Dynamics of Endogenous Population Growth in a Renewable Resource-Based Growth Model," *Ecological Economics*, 56, 256-267

LUCAS, R. E., JR. (1988): "On the Mechanics of Economic Development," *Journal of Monetary Economics* 22, 3-42.

MCCANDLESS, G. T. JR. AND N. WALLACE (1991): *Introduction to Dynamic Macroeconomic Theory: An Overlapping Generations Approach*, Harvard University

- Press, Cambridge, Massachusetts.
- MEADOWS S. et.al (1972), *The Limits to Growth*, Potomac Associates University book, New York.
- MITRA, T. (1980): "On Optimal Depletion of Exhaustible Resource: Existence and Characterization Results," *Econometrica* 48, 1431-1450.
- MOURMOURAS, A. (1993): "Conservationist Government Policies and Intergenerational Equity in an Overlapping Generations Model with Renewable Resources," *Journal of Public Economics*, 51, 249-268.
- NERLOVE, M. (1993): "Procreation, Fishing and Hunting: a Parable of Firewood and Other Tales," *American Journal of Agricultural Economics* 107, 1334-1357.
- NERLOVE, M. AND L. K. RAULT (1997): "Growth Models with Endogenous Population: A General Framework," in *Handbook of Population and Family Economics*.
- NERLOVE, M., A. RAZIN, AND E. SADKA (1986): "Some Welfare Theoretic Implications of Endogenous Fertility," *International Economic Review* 27, 3-31.
- _____ (1987): "*Household and Economy: Welfare Economics and Endogenous Fertility*," Academic Press, Boston, MA.
- OLSON, L. AND K. C. KNAPP (1997): "Exhaustible Resource Allocation in an Overlapping Generations Economy," *Journal of Environmental Economics and Management* 32, 277-292.
- RAZIN, A. AND U. BEN-ZION (1975): "An Intergenerational Model of Population Growth," *American Economic Review* 65, 923-933.
- ROBINSON, J. A. AND T. N. SRINIVASAN (1997): "Long-Term Consequences of Population Growth: Technological Change, Natural Resources, and the Environment," in *Handbook of Population and Family Economics*, Rosenzweig and O. Stark eds., Elsevier Science B.V.
- SAMUELSON, P. A. (1975): "The Optimal Growth Rate for Population," *International Economic Review* 16, 531-538.
- STIGLITZ, J. (1974): "Growth with Exhaustible Natural Resources: The Competitive Economy," *Review of Economic Studies Symposium*, 139-142.
- TAMURA, R. (2000): "Growth, Fertility and Human Capital: A Survey," *Spanish Economic Review*, 183-229.
- TAHVONEN, O. AND S. SALO (2001): "Economic Growth and Transition Between Renewable and Non-Renewable Energy Resources," *European Economic Review*, 45,

1379-1398.

TIROLE, J. (1985): "Asset Bubbles and Overlapping Generations," *Econometrica* 53, 1071-1100.

VALENTE, S. (2006): "Intergenerational Transfers, Lifetime Welfare and Resource Preservation," *Center of Economic Research, ETH Zurich, Working Paper 06/55*.

WENDNER, R. (2003): "Existence, Uniqueness, and Stability of Equilibrium in an OLG Economy", *Economic Theory*, 23, 165-174.

ADDENDUM

PROOFS OF PROPOSITION 3 AND LEMMA 4

Existence of Competitive Equilibrium in an Overlapping Generations Model with Exhaustible Resources, Technology Transition, and Endogenous Fertility

The presence of an exhaustible resource whose stock declines through time makes the problem non-stationary, and the existence of a competitive equilibrium becomes problematic. Furthermore, the equilibrium can no longer be computed recursively. This addendum provides a proof of Proposition 3, which asserts the existence of a competitive equilibrium for an economy endowed with a stock of fossil fuels, and a more detailed proof of Lemma 4. The addendum offers a contribution to the sparse literature on the existence of competitive equilibrium for overlapping-generations models due to Balasko and Shell (1980, 1981a, 1981b), Balasko, Cass, and Shell (1980), Wilson (1981), Galor and Ryder (1989).

In their efforts to generalize the finite-economy model of Arrow and Debreu to the overlapping-generations model, Balasko and Shell, *op cit.*, and Balako, Cass, and Shell, *op cit.*, have qualified the latter as one with double infinities: an infinity of consumers and an infinity of commodities. The infinity of consumers involves the infinite number of successive generations, while the infinity of commodities involve the infinite number of dated commodities – one commodity for each time period. These researchers established the existence of a competitive equilibrium for a simple overlapping-generations model of an exchange economy by showing – with the help of the Tychonoff theorem – that the competitive equilibrium for the overlapping-generations model is the limit of a sequence of truncated economies. The overlapping-generations model we formulate has both capital and an exhaustible resource. It also has an endogenous population structure because fertility decisions are determined by the maximizing behavior of successive young generations. Compared to the models of these authors, ours is much more complex, and it is not possible to invoke their results to assert that it has a competitive equilibrium.

Our existence proof proceeds in three stages. First, we show that when the economy is truncated at the end of a finite number of periods, the economy thus obtained has a competitive equilibrium. Next, we deploy a series of technical arguments to establish some upper and lower bounds on the prices, the birthrate, and the energy endowment per young individual in each period that apply uniformly across truncated economies. In the third stage, we use these bounds to show that a sequence of competitive equilibria – one competitive equilibrium for each truncated economy – has a subsequence that converges in the product topology of a denumerable family of finite-dimensional Euclidean spaces, and the limit of the subsequence is a competitive equilibrium of the infinite time horizon economy. The technique we employ in establishing convergence is Cantor’s famous diagonal trick used in the proof of the following version of the Tychonoff theorem: “*The product of a denumerable family of compact metrizable spaces is compact and metrizable.*” The interested reader can consult Dieudonné (1976, (12.5.9)).

We would like to point out the advantage of the existence proof technique based on the Tychonoff theorem over that based on the monotone mapping theorem (see, for example, Stokey, Lucas, and Prescott (1989) or Olson and Knapp, op cit.) that is often used to establish the existence of a competitive equilibrium for simple macroeconomic models with one state variable and formulated under the overlapping-generations framework. In the latter technique, it is necessary to establish first that the operator a fixed point of which constitutes a competitive equilibrium of the overlapping-generations model has some desired monotonicity property, and this is hard to show, especially when the model has several state variables, as is the case of our model. Furthermore, unlike the proof of Balasko and Shell, op cit., and Balasko, Cass, and Shell, op cit., our existence proof is accessible to readers with only a rudimentary knowledge of real analysis.

Suppose that the economy begins at time 0 in state (X_0, K_0, N_0^0, N_0^1) , with $X_0 > 0$, $K_0 \geq 0$. Let $\xi_0 = X_0 / N_0^0$ and $\kappa_0 = K_0 / N_0^0$ denote the initial oil endowment per young individual and the initial capital labor ratio, respectively. The initial energy endowments

per young individual are thus equal to $\xi_0 + \kappa_0$. Also, recall the assumption that $\xi_0 + \kappa_0 > e^{\min}(\delta)$, which ensures that the population does not collapse in finite time.

A.1. THE EXISTENCE OF COMPETITIVE EQUILIBRIUM FOR A TRUNCATED ECONOMY

Let T be a non-negative integer. If we truncate our economy at the end of period T , then we obtain an economy with a finite time horizon that we call the truncated economy with time horizon T . A price system for the truncated economy with time horizon T is a finite sequence $\mathfrak{P}^T = (\phi_t, \varphi_t, \omega_t, \rho_t)_{t=0}^T$. An allocation induced by \mathfrak{P}^T is a list of finite sequences

$$\mathfrak{Y}^T = (c_0^1, (c_t^0, c_{t+1}^1, b_t, x_{t+1}, k_{t+1})_{t=0}^{T-1}, (Q_t, S_t, L_t, Y_t)_{t=0}^T, (K_t^\#, S_t^\#)_{t=0}^T, (X_t, K_t, N_t^0, N_t^1)_{t=0}^T, c_T^0)$$

with the following properties: When the price system \mathfrak{P}^T prevails,

- (i) $c_0^1 = [\phi_0 X_0 + (1 - \delta + \rho_0)K_0] / N_0^1$,
- (ii) $(c_t^0, c_{t+1}^1, b_t, x_{t+1}, k_{t+1})$ is the optimal lifetime plan for a young individual of period t , ($t = 0, \dots, T - 1$),
- (iii) (Q_t, S_t, L_t, Y_t) is an optimal production plan of the representative firm in the consumption good sector in period t , ($t = 0, \dots, T$),
- (iv) $(K_t^\#, S_t^\#)$ is an optimal production plan for the representative producer of solar energy in period t , ($t = 0, \dots, T$),
- (v) $(X_t, K_t, N_t^0, N_t^1) = N_{t-1}^0(x_t, k_t, b_{t-1}, 1)$, ($t = 1, \dots, T$),
- (vi) $c_T^0 = \omega_T$.

Observe that (vi) represents the consumption of a young individual of period T . Because the problem ends at the end of period T , a young individual of this period has no future to plan for, and thus will neither save nor raise children; she will consume all the wages she earns. The pair $(\mathfrak{P}^T, \mathfrak{Y}^T)$ is said to constitute a *competitive equilibrium for the truncated economy with time horizon T* if the following market-clearing conditions are satisfied:

- (vii) $X_{t+1} + Q_t = X_t$, $0 < t < T$, and $X_T = Q_T$.

- (viii) $S_t = S_t^\#, 0 \leq t \leq T,$
- (ix) $L_t = N_t^0, 0 \leq t \leq T,$
- (x) $K_t^\# = K_t, 0 \leq t \leq T,$
- (xi) $[\phi_t X_t + (1 - \delta + \rho_t)K_t] + [N_t^0(c_t^0 + hb_t + k_{t+1})] = Y_t + (1 - \delta)K_t, 0 \leq t < T,$
 $[\phi_T X_T + (1 - \delta + \rho_T)K_T] + [N_T^0 c_T^0] = Y_T + (1 - \delta)K_T.$

The following proposition asserts the existence of a competitive equilibrium for a truncated economy, and gives some of its properties. In this proposition, we have let $\kappa_t = K_t / N_t^0$ and $q_t = Q_t / N_t^0$ denote, respectively, the equilibrium capital labor ratio and the equilibrium oil input per worker – both in period t .

PROPOSITION A.1: *Consider an economy with a positive stock of oil, and possibly a positive stock of capital. For any integer $T \geq 0$, the truncated economy with time horizon*

T has a competitive equilibrium, say $(\mathfrak{P}^T, \mathfrak{Y}^T)$, with $\mathfrak{P}^T = (\phi_t, \varphi_t, \omega_t, \rho_t)_{t=0}^T$ and

$$\mathfrak{Y}^T = (c_0^1, (c_t^0, c_{t+1}^1, b_t, x_{t+1}, k_{t+1})_{t=0}^{T-1}, (Q_t, S_t, L_t, Y_t)_{t=0}^T, (K_t^\#, S_t^\#)_{t=0}^T, (X_t, K_t, N_t^0, N_t^1)_{t=0}^T, c_T^0).$$

Under such a competitive equilibrium, the birthrate in each period before the last period is positive, and the following relationship holds between the price of oil and the price of renewable energy:

$$(A.1.1) \quad 0 < \min\{\phi_t, \rho_t\} = f'(q_t + \kappa_t) = \rho_t \leq \phi_t < \rho^{\max}(\delta), \quad (t = 0, \dots, T),$$

with equality holding for the second inequality in (A.1.1) if $q_t > 0$.

The existence of a competitive equilibrium for a truncated economy, and the fact that all equilibrium prices are positive can be established by using the well-known proof technique developed by Debreu, Gale, and Nikaido.⁶ Because fertility choice is endogenous, the number of consumers – old and young individuals – in each period is also endogenous, which means that the Debreu-Gale-Nikaido proof technique must be slightly modified to take into account the endogenous market size in each period. To see why the birthrate is positive in each of the periods before the last period, note that a zero

⁶ See Nikaido (1970, Chapter 10).

birthrate in any period $t < T$ implies a zero labor supply in the following period, leading to a positive excess demand for labor in that period, and this cannot arise in equilibrium. As for (A.1.1), it asserts that the equilibrium price of energy in each period is given by the equilibrium price of renewable energy in that period, and is always less than or equal to the equilibrium price of oil. To see why, note that in any period, if the stock of capital is positive, then the supply of renewable energy is positive, and its price must adjust to clear the market. Hence $\rho_t \leq \phi_t$. On the other hand, if the stock of capital is 0, then only oil is used in the production of the consumption good, and this implies $\rho_t \geq \phi_t$. In this case, we can set $\rho_t = \phi_t$ without changing the decisions taken by the representative firms in the oil and renewable energy markets as well as the lifetime plan chosen by a young individual of the preceding period. Also, the price of energy in each period before the last period must be lower than the critical level $\rho^{\max}(\delta)$; otherwise, the birthrate in this period will be 0, and we have just argued that this cannot arise in equilibrium.

A.2. UNIFORM BOUNDS ON TRUNCATED ECONOMIES

The following lemma establishes lower and upper bounds in two arbitrary successive periods that apply uniformly to all truncated economies.

LEMMA A.2: *Consider an arbitrary truncated economy in which ξ_t is the oil endowment per young individual, and κ_t is the capital labor ratio – both in period t . Also, suppose that ξ_t and κ_t are constrained to satisfy the following condition:*

$$(A.2.1) \quad (\xi_t + \kappa_t) \in [\underline{\eta}_t, \bar{\eta}_t],$$

where $\underline{\eta}_t$ and $\bar{\eta}_t$ are two constants satisfying $e^{\min}(\delta) < \underline{\eta}_t < \bar{\eta}_t$. Next, let $(\phi_t, \rho_t, \omega_t)$ be the price system and b_t be the birthrate – both in period t – under a competitive equilibrium of a truncated economy. Also, let ξ_{t+1} and κ_{t+1} denote, respectively, the oil endowment per young individual and the capital labor ratio – both in period $t+1$ – under this competitive equilibrium. We have the following results that hold uniformly for

all truncated economies with an energy endowment per young individual in period t that is constrained to satisfy (A.2.1) and all competitive equilibria of these economies.

- (i) ϕ_t is bounded below by $\underline{\phi}_t = f'(\bar{\eta}_t) > 0$ and above by a number, say $\bar{\phi}_t$.
- (ii) ρ_t is bounded below by $\underline{\rho}_t = f'(\bar{\eta}_t) > 0$ and above by a number, say $\bar{\rho}_t < \rho^{\max}(\delta)$.
- (iii) ω_t is bounded below by a number $\underline{\omega}_t > \omega^{\min}(\delta)$ and above by $\bar{\omega}_t = f(\bar{\eta}_t) - \bar{\eta}_t f'(\bar{\eta}_t)$.
- (iv) b_t is bounded below by a number $\underline{b}_t > 0$ and above by a number $\bar{b}_t < b^{\max}$.
- (v) $\xi_{t+1} + \kappa_{t+1}$ is bounded below by a number $\underline{\eta}_{t+1} > e^{\min}(\delta)$ and above by a number, say $\bar{\eta}_{t+1}$.

PROOF: (i) First, note that the equilibrium price of oil and the equilibrium price of renewable energy – both in period t – are bounded below by $\underline{\phi}_t = \underline{\rho}_t = f'(\bar{\eta}_t)$, uniformly for all $(\xi_t + \kappa_t) \in [\underline{\eta}_t, \bar{\eta}_t]$ and all equilibria of all truncated economies that satisfy the preceding constraint. Next, note that according to (A.1.1), if $\phi_t \geq \rho^{\max}(\delta)$, then oil will not be used as part of the energy inputs used in the production of the consumption good, and the entire oil endowment of the economy will be bought by the young generation of period t . Hence if the price of oil in period t is too high in a truncated economy, the value of the oil stock will exceed the labor income of the young individuals of period t , and this cannot happen in equilibrium. We shall let $\bar{\phi}_t$ denote a uniform upper bound for the equilibrium price of oil in period t . Property (i) of Lemma A.2 is now established.

(ii) Because we have exhibited a uniform lower bound for the price of renewable energy in period t , to prove (ii), we now only need to show the existence of a uniform upper bound $\bar{\rho}_t < \rho^{\max}(\delta)$ for the price of renewable energy in period t . To this end, suppose the contrary. Then for each positive integer n , we can find a positive integer $T(n)$ and a competitive equilibrium of the truncated economy with time horizon $T(n)$ such that

$$(A.2.2) \quad \rho^{\max}(\delta) - 1/n < f'(q_t(n) + \kappa_t(n)) < \rho^{\max}(\delta),$$

where $q_t(n)$ and $\kappa_t(n)$ represent, respectively, the oil input and the renewable energy input per worker in period t under the competitive equilibrium in question. Furthermore, if $\xi_t(n)$ and $\kappa_t(n)$ denote, respectively, the oil endowment per young individual and the capital labor ratio in period t under such an equilibrium, then the oil investment made by a young individual of period t is given by

$$\xi_t(n) - q_t(n) = [\xi_t(n) + \kappa_t(n)] - [q_t(n) + \kappa_t(n)],$$

and in the limit, we have

$$(A.2.3) \quad \lim_{n \rightarrow \infty} [\xi_t(n) - q_t(n)] \geq \underline{\eta}_t - e^{\min}(\delta) > 0.$$

We shall now show that (A.2.3) cannot hold in equilibrium. To this end, let

$$(c_t^0(n), c_{t+1}^1(n), b_t(n), x_{t+1}(n), k_{t+1}(n))$$

be the lifetime plan chosen by a young individual of period t . According to (A.2.2), when n is large, the equilibrium wage rate in period t will be in a small right neighborhood of $\omega^{\min}(\delta)$, and this induces a very low birthrate in period t ; that is, $b_t(n) \rightarrow 0$ when $n \rightarrow \infty$. Furthermore, because the equilibrium wage rate in period t is bounded above by $f(\bar{\eta}_t) - \bar{\eta}_t f'(\bar{\eta}_t)$, the capital investment made by such an individual must also be bounded above by $f(\bar{\eta}_t) - \bar{\eta}_t f'(\bar{\eta}_t)$. Hence the total energy input used by all the offspring of a young individual of period t must be bounded above by

$$x_{t+1}(n) + k_{t+1}(n) < \xi_t + k_{t+1}(n) < \bar{\eta}_t + f(\bar{\eta}_t) - \bar{\eta}_t f'(\bar{\eta}_t).$$

The total output of the consumption good produced by all the offspring of a young individual of period t is then bounded above by $F(\bar{\eta}_t + f(\bar{\eta}_t) - \bar{\eta}_t f'(\bar{\eta}_t), b_t(n))$, which tends to 0 when n tends to infinity. Finally, because the representative firm in the consumption good sector pays for the oil input in period $t + 1$ from part of its output, and the young generation of period $t + 1$ pays for its oil investment from its labor income, and labor income constitutes only a fraction of the output, the oil investment represented by the left side of the first inequality in (A.2.3) will yield a zero rate of return, and thus contributes nothing to the old-age consumption of a young individual of period t ; that is,

in the limit it is not rational at all for a young individual of period t to invest in oil, We have just established (ii) of Lemma A.2.

(iii) Property (iii) of Lemma A.2. follows directly from property (ii).

(iv): First note that if b_t is close to the saturation level b^{\max} , then using the first-order condition that characterizes the optimal lifetime plan of a young individual of period t we can assert that her current consumption and a fortiori her labor income will be large, contradicting (iii) of the lemma. This show the existence of an upper bound $\bar{b}_t < b^{\max}$. To show the existence of a positive lower bound for b_t , suppose it is not true. Then for each positive integer n , we can find a positive integer $T(n)$ and a competitive equilibrium of the truncated economy with time horizon $T(n)$ such that $b_t(n) < 1/n$, where $b_t(n)$ represents the equilibrium birthrate in period t under the competitive equilibrium in question. Because $b_t(n) \rightarrow 0$ when $n \rightarrow \infty$, the oil investment of a young individual of period t – as already shown in the proof of (ii) – will tend to 0 when $n \rightarrow \infty$. There are two possibilities to consider: $\delta < 1$ and $\delta = 1$.

If $\delta < 1$, then investing in capital will yield a rate of return of at least $1 - \delta > 0$, and this means – due to the Inada condition on the sub-utility function of consumption that in the limit the capital investment of a young individual of period t will be strictly positive. Furthermore, because the total output of the consumption good produced by all the offspring of a young individual of period t will be equal to 0 in the limit, the capital income that such an individual obtains by renting her capital in her old age will be 0. Hence in the limit, the rate of return to capital investment obtained by a young individual of period t will be $\lim_{n \rightarrow \infty} r_{t+1}(n) = 1 - \delta$. However, for a young individual of period t , at the rate of return to capital investment of $1 - \delta$, the number of offspring she chooses to raise will rise above 0 when the wage rate rises above the critical level $\omega^{\min}(\delta)$. Using (iii), we can then assert that in the limit the number of offspring raised by a young

individual of period t will be positive, contradicting the premise of the reductio ad absurdum, and (iv) is proved for the case capital does not depreciate completely.

If $\delta = 1$, then investing in capital also yields a zero rate of return, and a young individual of period t will not save, and in the limit she will spend her entire labor income on current consumption and raising children. The last result together with the fact $\omega_t > \underline{\omega}_t > \omega^{\min}(\delta)$ imply that in the limit the equilibrium birthrate in period t will be strictly positive, contradicting the premise of the reductio ad absurdum argument. Property (iv) of Lemma A.2 is now established.

(v) If $\xi_{t+1} + \kappa_{t+1}$ is not bounded below by a number $\underline{\eta}_{t+1} > e^{\min}(\delta)$, then for each positive integer n , we can find a positive integer $T(n)$ and a competitive equilibrium of the truncated economy with time horizon $T(n)$ such that

$$(A.2.4) \quad \rho^{\max}(\delta) - 1/n < f'(\xi_{t+1}(n) + \kappa_{t+1}(n)) < \rho^{\max}(\delta),$$

where $\xi_{t+1}(n)$ represents the equilibrium oil endowment per young individual, and $\kappa_{t+1}(n)$ the equilibrium capital labor ratio – both in period $t+1$. According to (A.1.1), $f'(q_{t+1}(n) + \kappa_{t+1}(n)) < \rho^{\max}(\delta)$. Using this result in (A.2.4), we can write

$$(A.2.5) \quad \rho^{\max}(\delta) - 1/n < f'(\xi_{t+1}(n) + \kappa_{t+1}(n)) \leq f'(q_{t+1}(n) + \kappa_{t+1}(n)) < \rho^{\max}(\delta).$$

It follows from (A.2.5) that $\omega_{t+1}(n) \downarrow \omega^{\min}(\delta)$ when $n \rightarrow \infty$, with the ensuing result that $\lim_{t+1}(n) = 0$, which contradicts the fact that the birthrate in period t , according to (iv), is bounded below uniformly for all equilibria of all truncated economies.

If $\xi_{t+1} + \kappa_{t+1}$ is not bounded above, then for each positive number n we can find a positive integer $T(n)$ and a competitive equilibrium for the truncated economy with time horizon $T(n)$ such that $\xi_{t+1}(n) + \kappa_{t+1}(n) > n$, where $\xi_{t+1}(n)$ represents the equilibrium oil endowment per young individual, and $\kappa_{t+1}(n)$ the equilibrium capital labor ratio – both in period $t+1$. Hence either $\xi_{t+1}(n)$ or $\kappa_{t+1}(n)$ must be large when n is large. If $\xi_{t+1}(n)$ is large, then using the result already proven that the birthrate of a young individual of

period t is bounded below uniformly for all equilibria of all truncated economies with time horizon $T(n)$, we can assert that the amount of oil bought by a young individual of period t , namely $b_t(n)\xi_{t+1}(n)$, will exceed ξ_t , the oil endowment per worker in period t , which is clearly not possible. On the other hand, if $\kappa_{t+1}(n)$ is large when n is large, then because the wage rate in period t is bounded above uniformly for all equilibria of all truncated economies, the capital investment of a young individual of period t will exceed her labor income when n is large, and this is not possible. The second part of (v) is now established. ■

A.3. EXISTENCE OF COMPETITIVE EQUILIBRIUM FOR THE INFINITE TIME HORIZON ECONOMY WITH FOSSIL FUELS

For each integer $T = 1, 2, \dots$, let $(\mathfrak{P}^T, \mathfrak{V}^T)$, where $\mathfrak{P}^T = (\phi_t^T, \varphi_t^T, \rho_t^T, \omega_t^T)_{t=0}^T$ and

$$\mathfrak{V}^T = \left(c_0^{1,T}, (c_t^{0,T}, c_{t+1}^{1,T}, b_t^T, x_{t+1}^T, k_{t+1}^T)_{t=0}^{T-1}, (Q_t^T, S_t^T, L_t^T, Y_t^T)_{t=0}^T, \right. \\ \left. (K_t^{\#,T}, S_t^{\#,T})_{t=0}^T, (X_t^T, K_t^T, N_t^{0,T}, N_t^{1,T})_{t=0}^T, c_T^{0,T} \right)$$

be a competitive equilibrium for the truncated economy with time horizon T . We shall now show that the sequence $(\mathfrak{P}^T, \mathfrak{V}^T)_{T=1}^\infty$ has a subsequence that converges in the product topology of a denumerable family of Euclidean spaces, and that the limit of this subsequence is a competitive equilibrium for the infinite time horizon economy.

Applying Lemma A.2 to ξ_0 and κ_0 , we can assert the existence of the following bounds for the price system in period 0 that apply uniformly for all the elements of the sequence

$$(\mathfrak{P}^T, \mathfrak{V}^T)_{T=1}^\infty:$$

$$(A.3.1) \quad 0 < \underline{\phi}_0 \leq \phi_0^T \leq \bar{\phi}_0,$$

$$(A.3.2) \quad 0 < \underline{\rho}_0 \leq \rho_0^T \leq \bar{\rho}_0 < \rho^{\max}(\delta),$$

$$(A.3.3) \quad \omega^{\min}(\delta) < \underline{\omega}_0 \leq \omega_0^T \leq \bar{\omega}_0.$$

Furthermore, there exists a lower bound $\underline{b}_0 > 0$ on the equilibrium birthrate in period 0, such that

$$(A.3.4) \quad 0 < \underline{b}_0 \leq b_0^T \leq \bar{b}_0 < b^{\max}$$

holds uniformly for all the elements of the sequence $(\mathcal{R}^T, \mathcal{Y}^T)_{T=1}^\infty$. Also, there exist bounds $\underline{\eta}_1$ and $\bar{\eta}_1$, such that

$$(A.3.5) \quad e^{\min}(\delta) < \underline{\eta}_1 \leq \xi_1^T + \kappa_1^T \leq \bar{\eta}_1$$

holds uniformly for all the elements of the sequence $(\mathcal{R}^T, \mathcal{Y}^T)_{T=1}^\infty$.

The procedure used to obtain the bounds on the prices and the birthrate in period 0 as well as the bounds on the energy endowment per young individual of period 1 can be repeated ad infinitum to obtain a sequence $(\underline{\phi}_t, \bar{\phi}_t, \underline{\rho}_t, \bar{\rho}_t, \underline{\omega}_t, \bar{\omega}_t, \underline{b}_t, \bar{b}_t, \underline{\eta}_{t+1}, \bar{\eta}_{t+1})_{t=0}^\infty$, such that for $t = 0, 1, \dots$,

$$(A.3.6) \quad 0 < \underline{\phi}_t \leq \phi_t^T \leq \bar{\phi}_t,$$

$$(A.3.7) \quad \underline{\rho}_t \leq \rho_t^T \leq \bar{\rho}_t < \rho^{\max}(\delta),$$

$$(A.3.8) \quad \omega^{\min}(\delta) < \underline{\omega}_t \leq \omega_t^T \leq \bar{\omega}_t.$$

$$(A.3.9) \quad 0 < \underline{b}_t \leq b_t^T \leq \bar{b}_t < b^{\max},$$

and

$$(A.3.10) \quad e^{\min}(\delta) < \underline{\eta}_{t+1} \leq \xi_{t+1}^T + \kappa_{t+1}^T \leq \bar{\eta}_{t+1}$$

hold uniformly for all the elements of the sequence $(\mathcal{R}^T, \mathcal{Y}^T)_{T=1}^\infty$, with $t < T - 1$.

Now for each integer t , let Z_t denote the set of integers greater than or equal to t . Invoking the Bolzano-Weierstrass theorem, we can assert the existence of an increasing map $\tau_0 : n \rightarrow \tau_0(n)$ of Z_1 into itself, such that the sequence

$$\left(\phi_0^{\tau_0(n)}, \rho_0^{\tau_0(n)}, \omega_0^{\tau_0(n)}, b_0^{\tau_0(n)}, \xi_1^{\tau_0(n)}, \kappa_1^{\tau_0(n)} \right)_{n=1}^\infty$$

has a limit, say $(\phi_0, \rho_0, \omega_0, b_0, \xi_1, \kappa_1)$. The process just described can be repeated ad infinitum to obtain a sequence of increasing maps $\tau_0, \tau_1, \tau_2, \dots$ with

$$\text{domain}(\tau_t) = \text{range}(\tau_t) = \text{image}(\tau_{t-1}) \cap Z_{t+1}, \quad (t = 1, 2, \dots),$$

such that for each $t = 0, 1, \dots$, we have

$$\lim_{n \rightarrow +\infty} \left(\begin{array}{c} \phi_{t'}^{\tau_t(\dots\tau_1(\tau_0(n)))}, \rho_{t'}^{\tau_t(\dots\tau_1(\tau_0(n)))}, \omega_{t'}^{\tau_t(\dots\tau_1(\tau_0(n)))}, \\ b_{t'}^{\tau_t(\dots\tau_1(\tau_0(n)))}, \xi_{t'+1}^{\tau_t(\dots\tau_1(\tau_0(n)))}, \kappa_{t'+1}^{\tau_t(\dots\tau_1(\tau_0(n)))} \end{array} \right)_{t'=0}^t = (\phi_{t'}, \rho_{t'}, \omega_{t'}, b_{t'}, \xi_{t'+1}, \kappa_{t'+1})_{t'=0}^t.$$

Let $\mathcal{P} = (\phi_t, \rho_t, \omega_t)_{t=0}^\infty$. Now for each $t = 0, 1, \dots$, the convergence of the price system, the convergence of the birthrate – all in period t – and the convergence in period $t + 1$ of the oil endowment per young individual as well as the capital labor ratio imply the convergence of the lifetime plan of a young individual of period t ; that is, for each t ,

$$(A.3.11) \quad \lim_{n \rightarrow +\infty} \left(c_t^{0, \tau_t(\dots\tau_1(\tau_0(n)))}, c_{t+1}^{1, \tau_t(\dots\tau_1(\tau_0(n)))}, b_t^{\tau_t(\dots\tau_1(\tau_0(n)))}, x_{t+1}^{\tau_t(\dots\tau_1(\tau_0(n)))}, k_{t+1}^{\tau_t(\dots\tau_1(\tau_0(n)))} \right) = (c_t^0, c_{t+1}^1, b_t, x_{t+1}, k_{t+1})$$

exists. In particular, the convergence of the price of oil and the rental rate of capital imply the convergence of the consumption of an old individual in period 0; that is,

$$\lim_{n \rightarrow +\infty} c_0^{1, \tau_1(\dots\tau_1(\tau_0(n)))} = c_0^1$$

exists. By continuity, (A.3.11) is the lifetime plan that maximizes the utility of a young individual of period t under the price system \mathcal{P} .

Using the convergence of the lifetime plans of the successive young generations, we can assert the convergence of the state of the economy in each period; that is,

$$\lim_{n \rightarrow +\infty} \left(X_t^{\tau_t(\dots\tau_1(\tau_0(n)))}, K_t^{\tau_t(\dots\tau_1(\tau_0(n)))}, N_t^{0, \tau_t(\dots\tau_1(\tau_0(n)))}, N_t^{1, \tau_t(\dots\tau_1(\tau_0(n)))} \right) = (X_t, K_t, N_t^0, N_t^1)$$

exists.

The market-clearing conditions in each period imply that the production plan of the representative firm that produces the consumption good in each period and the production plan of the representative firm in the backstop sector all converge; that is

$$(A.3.12) \quad \lim_{n \rightarrow +\infty} \left(Q_t^{\tau_t(\dots\tau_1(\tau_0(n)))}, S_t^{\tau_t(\dots\tau_1(\tau_0(n)))}, L_t^{\tau_t(\dots\tau_1(\tau_0(n)))}, Y_t^{\tau_t(\dots\tau_1(\tau_0(n)))} \right) = (Q_t, S_t, L_t, Y_t)$$

and

$$(A.3.13) \quad \lim_{n \rightarrow +\infty} \left(K_t^{\#, \tau_t(\dots\tau_1(\tau_0(n)))}, S_t^{\#, \tau_t(\dots\tau_1(\tau_0(n)))} \right) = (K_t^\#, S_t^\#)$$

both exist. By continuity, (A.3.12) and (A.3.13) are profit-maximizing plans, respectively, of the representative firm producing the consumption good and the representative firm producing renewable energy in period t under the price system \mathcal{P} .

Let

$$\mathfrak{P} = \left(c_0^1, (c_t^0, c_{t+1}^1, b_t, x_{t+1}, k_{t+1})_{t=0}^\infty, (Q_t, S_t, L_t, Y_t)_{t=0}^\infty, (K_t^\#, S_t^\#)_{t=0}^\infty, (X_t, K_t, N_t^0, N_t^1)_{t=0}^\infty \right)$$

We have just established the following proposition, which is Proposition 3 in the main body of the paper.

PROPOSITION A.3: *Consider an economy with a positive stock of oil and possibly a positive stock of backstop capital. The pair $(\mathfrak{P}, \mathfrak{P})$, thus constructed, constitutes a competitive equilibrium of this economy. Under such a competitive equilibrium, the birthrate in each period is positive. Furthermore, the following relationship holds between the price of oil and the price of renewable energy:*

$$(A.3.14) \quad 0 < \min\{\phi_t, \rho_t\} = f'(q_t + \kappa_t) = \rho_t \leq \phi_t < \rho^{\max}(\delta), \quad (t = 0, 1, \dots),$$

with equality holding for the third inequality in (A.3.14) if $q_t > 0$.

In (A.3.14), we have let $\kappa_t = K_t / N_t^0$ and $q_t = Q_t / N_t^0$. Also, we note in passing that (A.3.14) holds for any competitive equilibrium, not just the one constructed in the proof of Proposition A.3.

A.4: PROOF OF LEMMA 4

According to Lemma 3.(ii) in the main body of the paper, if the initial energy endowment per young individual is large, then the energy endowment per young individual in period 1 will be lower, and this process will continue through time as long as the energy endowments per young individual in the subsequent periods are still large. While this happens, the price of energy will be rising. On the other hand, if the initial energy endowment per young individual is close to the critical level $e^{\min}(\delta)$, then the birthrate in period 0 will be extremely low, resulting in a large energy endowment per young individual in period 1, and from period 1 on, the price of energy will begin its climb, as just described. Thus, we can expect the existence of a positive lower bound for the price of energy that applies eventually to truncated economies with long time horizons. The

following lemma, which is Lemma 4 in the main body of the paper, confirms this intuition.

LEMMA A.4: *There exists a number $\rho^- > 0$ with the following property: If $\delta < \rho^-$, then for any value $\xi_0 + \kappa_0$ of the initial energy endowment per young individual, there exists a non-negative integer $t^-(\xi_0 + \kappa_0)$, such that in any truncated economy with time horizon of at least $t^-(\xi_0 + \kappa_0)$, the equilibrium price of energy satisfies the following condition:*

$$\rho_t = \min\{\phi_t, \rho_t\} = f'(q_t + \kappa_t) \geq \rho^-, \quad (t \geq t^-(\xi_0 + \kappa_0)).$$

We prove Lemma A.4 in a series of claims.

CLAIM 1: *For each value of $\delta, 0 \leq \delta \leq 1$, there exists a number $\varepsilon > 0$ such that the following result holds uniformly for all truncated economies: If the equilibrium energy endowment per young individual in period t is such that $f'(\xi_t + \kappa_t) < \rho^{\max}(\delta)$, then the equilibrium energy endowment per young individual in the next period satisfies the inequality $f'(\xi_{t+1} + \kappa_{t+1}) \leq \rho^{\max}(\delta) - \varepsilon$.*

PROOF: Indeed, if the claim is not true, then for each positive integer n , we can find a truncated economy with $\xi_t(n)$ as the equilibrium oil endowment per young individual and $\kappa_t(n)$ as the equilibrium capital labor ratio – both in period t , such that $f'(\xi_t(n) + \kappa_t(n)) < \rho^{\max}(\delta)$, but

$$(A.4.1) \quad \rho^{\max}(\delta) - 1/n < f'(\xi_{t+1}(n) + \kappa_{t+1}(n)).$$

Using (A.4.1), the fact $q_{t+1}(n) + \kappa_{t+1}(n) \leq \xi_{t+1}(n) + \kappa_{t+1}(n)$, and (A.3.14), we can write

$$(A.4.2) \quad \rho^{\max}(\delta) - 1/n < f'(\xi_{t+1}(n) + \kappa_{t+1}(n)) \leq f'(q_{t+1}(n) + \kappa_{t+1}(n)) < \rho^{\max}(\delta).$$

Because the sequence $f'(\xi_t(n) + \kappa_t(n)), n = 1, 2, \dots$, is bounded, it has a convergent subsequence that, by abuse of notation, we still denote by $f'(\xi_t(n) + \kappa_t(n)), n = 1, 2, \dots$. Let

$\rho = \liminf'(\xi_t(n) + \kappa_t(n))$. If $\rho = 0$, then $(\xi_t(n) + \kappa_t(n))$ will be large when n is large, which, according to Lemma 3.(ii), leads to a large value of $(\xi_{t+1}(n) + \kappa_{t+1}(n))$. A large value of $(\xi_{t+1}(n) + \kappa_{t+1}(n))$ in turn implies a small value of $f'(\xi_{t+1}(n) + \kappa_{t+1}(n))$, contradicting (A.4.1). Thus, we must have $\rho > 0$. If $\rho = \rho^{\max}(\delta)$, then when n is large, the equilibrium price of energy in period t will be in a small left neighborhood of $\rho^{\max}(\delta)$, which induces a very low value of $f'(\xi_{t+1}(n) + \kappa_{t+1}(n))$, again contradicting (A.4.1). Hence the premise of the reductio ad absurdum argument leads to the following result: $0 < \rho < \rho^{\max}(\delta)$. Now pick a large value of n , then apply Lemma A.2.(v) to $\xi_t(n) + \kappa_t(n)$ to assert that $\xi_{t+1}(n) + \kappa_{t+1}(n)$ is bounded below by a number greater than $e^{\min}(\delta)$, no matter how large n is; that is, when n is large, $f'(\xi_{t+1}(n) + \kappa_{t+1}(n))$ remains outside a left neighborhood of $\rho^{\max}(\delta)$, contradicting (A.4.2). ■

We shall let $\varepsilon^+(\delta)$ denote the supremum of the set of ε 's for which Claim 1 holds. Note that $\varepsilon^+(\delta) > 0$ for all $0 \leq \delta \leq 1$.

CLAIM 2: *For each value of $\delta, 0 \leq \delta \leq 1$, there exists a number $\varepsilon > 0$ such that the following results hold uniformly for all truncated economies:*

- (i) *If $f'(\xi_t + \kappa_t) < \varepsilon$, then $f'(\xi_t + \kappa_t) < f'(\xi_{t+1} + \kappa_{t+1})$.*
- (ii) *If $\varepsilon \leq f'(\xi_t + \kappa_t) \leq \rho^{\max}(\delta) - \varepsilon^+(\delta)$, then $\varepsilon \leq f'(\xi_{t+1} + \kappa_{t+1})$.*

PROOF: Indeed, if the claim is not true, then for each positive integer n , we can find a truncated economy, with $\xi_t(n)$ as the oil endowment per young individual and $\kappa_t(n)$ as the capital labor ratio – both in period t – such that either (i) or (ii) of the claim does not hold for $\varepsilon = 1/n$.

Next, note that the condition $f'(\xi_t(n) + \kappa_t(n)) < 1/n$ implies that $\xi_t(n) + \kappa_t(n)$ will be large when n is large, and thus according to Lemma 4.(ii), $\xi_{t+1}(n) + \kappa_{t+1}(n)$ will be lower than $\xi_t(n) + \kappa_t(n)$, which in turn implies $f'(\xi_t(n) + \kappa_t(n)) < f'(\xi_{t+1}(n) + \kappa_{t+1}(n))$. Hence

if the claim is not true, then only (ii) of the claim will be violated when n is large, i.e., for large n , we have $f'(\xi_{t+1}(n) + \kappa_{t+1}(n)) < 1/n$, which in turn implies

$$(A.4.3) \quad \lim(\xi_{t+1}(n) + \kappa_{t+1}(n)) = \infty.$$

Because

$$0 < f'(\xi_t(n) + \kappa_t(n)) \leq \rho^{\max}(\delta) - \varepsilon^+(\delta), n = 1, 2, \dots,$$

the sequence $f'(\xi_t(n) + \kappa_t(n)), n = 1, 2, \dots$, has a convergent subsequence that, by abuse of notation, we still denote by $f'(\xi_t(n) + \kappa_t(n)), n = 1, 2, \dots$. Let $\rho = \lim f'(\xi_t(n) + \kappa_t(n))$. We have $0 \leq \rho \leq \rho^{\max}(\delta) - \varepsilon^+(\delta)$. Furthermore, because (ii) of the claim is violated, we must have $\rho > 0$. Hence $\rho \in (0, \rho^{\max}(\delta) - \varepsilon^+(\delta)]$, and $\xi_t(n) + \kappa_t(n)$ will tend to $\kappa = [f']^{-1}(\rho) > e^{\min}(\delta)$. Now apply Lemma A.2 (v) to $\xi_t(n) + \kappa_t(n)$, we can assert that $\xi_{t+1}(n) + \kappa_{t+1}(n)$ is bounded above, and this contradicts (A.4.3). ■

We shall let $\varepsilon^-(\delta)$ be the supremum of the set of ε 's for which Claim 2 is true. Note that $\varepsilon^-(\delta) > 0$ for all $0 \leq \delta \leq 1$.

CLAIM 3: *There exists a number $\varepsilon^+(0_+) > 0$ such that $\varepsilon^+(\delta) > \varepsilon^+(0_+)$ when δ is small enough.*

PROOF: If the claim is not true, then for each positive integer n , there exists a rate of capital depreciation $\delta(n) < 1/n$, such that $\varepsilon^+(\delta(n)) < 1/n$. Let $\theta > 1$ be a given number, and consider the sequence $\theta\varepsilon^+(\delta(n)), n = 1, 2, \dots$. It follows from the definition of $\varepsilon^+(\delta(n))$ that there is a competitive equilibrium of a truncated economy such that $f'(\xi_t(n) + \kappa_t(n)) < \rho^{\max}(\delta(n))$, but

$$f'(\xi_{t+1}(n) + \kappa_{t+1}(n)) > \rho^{\max}(\delta(n)) - \theta\varepsilon^+(\delta(n)),$$

where $\xi_t(n)$ and $\kappa_t(n)$ denote, respectively, the oil endowment per young individual and the capital labor ratio – both in a particular period t . The preceding inequality together with the fact that

$$f'(\xi_{t+1}(n) + \kappa_{t+1}(n)) \leq f'(q_{t+1}(n) + \kappa_{t+1}(n)) < \rho^{\max}(\delta(n))$$

allow us to write

$$(A.4.4) \quad \begin{aligned} \rho^{\max}(\delta(n)) - \theta \varepsilon^+(\delta(n)) &< f'(\xi_{t+1}(n) + \kappa_{t+1}(n)) \\ &\leq f'(q_{t+1}(n) + \kappa_{t+1}(n)) < \rho^{\max}(\delta(n)). \end{aligned}$$

Hence when $n \rightarrow \infty$, we have

$$(A.4.5) \quad \liminf f'(\xi_{t+1}(n) + \kappa_{t+1}(n)) = \rho^{\max}(0).$$

Next, note that because the sequence $f'(\xi_t(n) + \kappa_t(n))$, $n = 1, 2, \dots$, is bounded, it has a convergent subsequence that we still denote by $f'(\xi_t(n) + \kappa_t(n))$, $n = 1, 2, \dots$. Let $\rho = \liminf f'(\xi_t(n) + \kappa_t(n))$. Because

$$f'(\xi_t(n) + \kappa_t(n)) \leq f'(q_t(n) + \kappa_t(n)) < \rho^{\max}(\delta(n)),$$

we have $0 \leq \rho \leq \rho^{\max}(0)$.

If $\rho = 0$, then $f'(\xi_t(n) + \kappa_t(n))$ is small – and a fortiori $(\xi_t(n) + \kappa_t(n))$ is large – when n is large. Applying Lemma 3.(ii), we can then assert that $(\xi_{t+1}(n) + \kappa_{t+1}(n))$ is large, i.e., $f'(\xi_{t+1}(n) + \kappa_{t+1}(n))$ is small when n is large, contradicting (A.4.5). If $\rho = \rho^{\max}(0)$, then when n is large, $(\xi_t(n) + \kappa_t(n))$ is close to $e^{\min}(\delta(n))$, which in turn implies a large savings offspring ratio generated by the maximizing behaviour of a young individual of period t . A high savings offspring ratio in period t means a large energy endowment per worker $\xi_{t+1}(n) + \kappa_{t+1}(n)$ in period $t + 1$, again contradicting (A.4.5). We have just shown that $0 < \rho < \rho^{\max}(0)$.

Now note that $0 < \rho < \rho^{\max}(0)$ implies $e^{\min}(0) < \kappa = [f']^{-1}(\rho)$. The argument used to prove Lemma A.2.(v) can also be used here to show that there is a lower bound for

$\xi_{t+1}(n) + \kappa_{t+1}(n)$ that is greater than $e^{\min}(0)$ and that applies uniformly for all large values of n , again contradicting (A.4.5). ■

CLAIM 4: *There exists a number $\rho^- > 0$, such that $\rho^- < \varepsilon^-(\delta)$ when δ is small enough.*

PROOF: If the claim is not true, then for each positive integer n , we can find a rate of capital depreciation $\delta(n) < 1/n$, such that $\varepsilon^-(\delta(n)) < 1/n$. Let $\theta > 1$ be a given number, and consider the sequence $\theta\varepsilon^-(\delta(n)), n = 1, 2, \dots$. It follows from the definition of $\varepsilon^-(\delta(n))$ that for each n , there exists a truncated economy with $\xi_0(n)$ as the initial oil endowment per young individual and $\kappa_0(n)$ as the initial capital labor ratio, such that

$$(A.4.6) \quad \theta\varepsilon^-(\delta(n)) \leq f'(\xi_0(n) + \kappa_0(n)) \leq \rho^{\max}(\delta(n)) - \varepsilon^+(\delta(n)),$$

but

$$(A.4.7) \quad f'(\xi_1(n) + \kappa_1(n)) < \theta\varepsilon^-(\delta(n)) \leq f'(\xi_0(n) + \kappa_0(n)).$$

It follows from the first inequality in (A.4.7) and the premise of the reductio ad absurdum argument $\varepsilon^-(\delta(n)) < 1/n$ that

$$(A.4.8) \quad \liminf f'(\xi_1(n) + \kappa_1(n)) = 0.$$

Because the sequence $f'(\xi_0(n) + \kappa_0(n)), n = 1, 2, \dots$, is bounded, it has a convergent subsequence that we still denote by $f'(\xi_0(n) + \kappa_0(n)), n = 1, 2, \dots$. Let $\rho = \liminf f'(\xi_0(n) + \kappa_0(n))$. We have

$$(A.4.9) \quad 0 \leq \rho \leq \rho^{\max}(0) - \varepsilon^+(0_+).$$

Note that the second inequality in (A.4.9) has been obtained by using (i) the second inequality in (A.4.6) in the limit, (ii) $\lim_{\delta \rightarrow 0} \rho^{\max}(\delta) = \rho^{\max}(0)$, and Claim 3. If $\rho = 0$, then $f'(\xi_0(n) + \kappa_0(n))$ is small – and a fortiori $(\xi_0(n) + \kappa_0(n))$ is large – when n is large. Applying Lemma 3.(ii), we can then assert that when n is large, we will have $\xi_1(n) + \kappa_1(n) < \xi_0(n) + \kappa_0(n)$, i.e., $f'(\xi_1(n) + \kappa_1(n)) > f'(\xi_0(n) + \kappa_0(n))$, and this last

result contradicts (A.4.7). We have just shown that $0 < \rho < \rho^{\max}(0)$ under the premise of the reductio ad absurdum argument.

Let $\kappa = [f']^{-1}(\rho)$ and $\omega = f(\kappa) - \kappa f'(\kappa)$. We have $\omega > \omega^{\min}(0)$. For a young individual whose labor income is ω and who obtains 1 as the rate of return to her savings, the number of offspring she raises is strictly positive, according to Lemma A.2.(iv). Hence by continuity, when n is large, the number of offspring raised by a young individual of period 0 under the competitive equilibrium associated with n will have a positive lower bound that applies uniformly to all n . This result together with the fact that κ is finite imply that the energy endowment per young individual in period 1 is bounded above, contradicting (A.4.8). Hence the claim is true. ■

We are now ready to prove Lemma A.4. Let δ , with $\delta < \rho^-$, be the rate of capital depreciation for the truncated economies. If the initial energy endowment per young individual $\xi_0 + \kappa_0$ belongs to the interval $[\varepsilon^-(\delta), \rho^{\max}(\delta) - \varepsilon^+(\delta)]$, then using the definition of $\varepsilon^-(\delta)$ and $\varepsilon^+(\delta)$, we can assert that

$$f'(\xi_t + \kappa_t) \in [\varepsilon^-(\delta), \rho^{\max}(\delta) - \varepsilon^+(\delta)], \quad (t \geq 0),$$

and we can set $t^-(\xi_0 + \kappa_0) = 0$. If $f'(\xi_0 + \kappa_0) \in [\varepsilon^-(\delta), \rho^{\max}(\delta) - \varepsilon^+(\delta)]$, then using the definition of $\varepsilon^-(\delta)$, we can assert that $f'(\xi_0 + \kappa_0) < f'(\xi_1 + \kappa_1)$. Furthermore, if

$$f'(\xi_1 + \kappa_1) \in [\varepsilon^-(\delta), \rho^{\max}(\delta) - \varepsilon^+(\delta)],$$

then we can set $t^-(\xi_0 + \kappa_0) = 1$. Otherwise, we can continue the process to obtain a chain of inequalities

$$f'(\xi_0 + \kappa_0) < f'(\xi_1 + \kappa_1) < f'(\xi_2 + \kappa_2) < \dots$$

If $f'(\xi_t + \kappa_t) \in [\varepsilon^-(\delta), \rho^{\max}(\delta) - \varepsilon^+(\delta)]$ for some finite t , then set $t^-(\xi_0 + \kappa_0) = t$.

Otherwise, we obtain a monotone increasing sequence $f'(\xi_t + \kappa_t), t = 0, 1, \dots$ having $\varepsilon^-(\delta)$ as the limit. Because $\rho^- < \varepsilon^-(\delta)$, we will have $f'(\xi_t + \kappa_t) > \rho^-$ in finite time, and we can set $t^-(\xi_0 + \kappa_0) = t$ for the first value of t this occurs. ■

REFERENCES

- BALASKO, Y. AND K. SHELL (1980): "The Overlapping-Generations Model, I: The Case of Pure Exchange without Money," *Journal of Economic Theory* 23, 281-306.
- _____ (1981a): "The Overlapping-Generations Model, II: The Case of Pure Exchange with Money," *Journal of Economic Theory* 24, 112-142.
- _____ (1981b): "The Overlapping-Generations Model, III: The Case of Log-Linear Utility Functions," *Journal of Economic Theory* 24, 143-152.
- BALASKO, Y., D. CASS, AND K. SHELL (1980): "Existence of Competitive Equilibrium in a General Overlapping-Generations Model," *Journal of Economic Theory* 23, 307- 322.
- DIEUDONNÉ, J. (1976): *Treatise on Analysis*, Volume II, Enlarged and Corrected Printing, Academic Press.
- GALOR, O. AND H. E. RYDER (1989): "Existence, Uniqueness, and Stability in Overlapping-Generation Model with Productive Capital," *Journal of Economic Theory*, 49, 360-375.
- OLSON, L. AND K. C. KNAPP (1997): "Exhaustible Resource Allocation in an Overlapping Generations Economy," *Journal of Environmental Economics and Management* 32, 277-292.
- STOKEY, N. L., ROBERT LUCAS, JR., AND EDWARD C. PRESSCOT (1989): "*Recursive Methods in Economic Dynamics*," Harvard University Press, Cambridge, Massachusetts.
- WILSON, CHARLES A. (1981): "Equilibrium in Dynamic Models with an Infinity of Agents," *Journal of Economic Theory* 24, 95-111.