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Abstract

There are two main approaches for defining social welfare relations for an economy with infinite horizon. The first one is to consider the set of intertemporal utility streams generated by a general set of bounded consumptions, and define a preference relation between them. This relation is ideally required to satisfy two main axioms, the Pareto axiom, which guarantees efficiency, and the Anonymity axiom, which guarantees equity. Basu and Mitra [2003] show that it is impossible to represent by a function a preference relation embodying both the efficiency and equity requirements and Basu and Mitra [2007] propose and characterize a new welfare criterion called utilitarian social welfare relation.

In the same framework, Chichilnisky [1996] proposes two axioms that capture the idea of sustainable growth: non-dictatorship of the present and non-dictatorship of the future, and exhibits a mixed criterion, adding a discounted utilitarian part (with possibly non constant discount rates), which gives a dictatorial role to the present, and a long term part, which gives a dictatorial role to the future. The drawback of Chichilnisky's approach is that it often does not allow to explicitly characterize optimal growth

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paths with optimal control techniques. Moreover, we observe that the optimal solution obtained with Chichilnisky's criterion, cannot in general be approximated by a sequence of optimal solutions with finite horizon.

Our aim is less general than Chichilnisky's, and Basu and Mitra's: we want to have a non-dictatorial criterion for optimal growth models. Instead of l_+^∞ as set of utilities, we just consider the set of utilities of consumptions which are generated by a specific technology. We show that the undiscounted utilitarian criterion pioneered by Ramsey [1928] is not only convenient if one wants to solve an optimal growth problem but also sustainable, efficient and equitable.

JEL classification codes: D60, D70, D90, Q0

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1 Introduction

Optimal growth theory is largely built around the discounted utilitarian approach, but the debate between authors subscribing to discounted utilitarianism and authors rejecting it has always been vivid. Ramsey [1928] himself, in a much-cited statement, held that "*discount later enjoyments in comparison with earlier ones [is] a practice which is ethically indefensible and arises merely from the weakness of imagination*". This debate is particularly passionate when the sustainability of growth is concerned, because the process of discounting forces a fundamental asymmetry between present and future generations, particularly those in the distant future, and so appears to be in contradiction with the intergenerational equity concern underlying the search for sustainability.

There are two main approaches for defining social welfare relations for an economy with infinite horizon. The first one is to consider the set of intertemporal utility streams generated by a general set of bounded consumptions, and define a preference relation between them. This relation is ideally required to satisfy two main axioms, the Pareto axiom and the Anonymity axiom. Pareto guarantees efficiency. Broadly speaking, it demands that the preference relation is sensitive to the well-being of each generation. Anonymity guarantees equity, that is an equal treatment of all generations. Besides, preferences are often required to be independent and to satisfy some continuity, transitivity and completeness properties. Unfortunately, in this framework, Basu and Mitra [2003] show that no social welfare function can satisfy simultaneously the Pareto and Anonymity axioms: it is impossible to represent by a function a preference relation embodying both the efficiency and equity requirements.

The second approach abandons the axiomatic foundations and proposes social welfare functions in the framework of optimal growth models. In this setting, we consider the consumption streams generated by a specific technology. Several welfare functions have been proposed in the literature for computing optimal sustainable growth paths[1]. The Rawlsian or maximin criterion (Rawls [1971], Solow [1974]) maximizes the utility of the least well off generation. The undiscounted utilitarian criterion (Ramsey [1928], Koopmans [1965]) minimizes the infinite sum of the differences between actual utilities and their upper bound (the bliss), with a zero utility discount rate. The overtaking criterion (von Weizacker [1965], Gale [1967]) replaces an infinite sum of utilities by a finite one, and says that one utility stream is better than another one if from some date on the first one is greater than the second one. Whereas the usual discounted utilitarian criterion assumes a constant discount rate, criteria with non constant and declining discount rates, hyperbolic for instance, have also been used (see

[1]For a complete survey on these criteria and their abilty to cope with the sustainability issue, see Heal [1998].

e.g. Heal [1998]).

Basu and Mitra [2007] propose and characterize a new welfare criterion called the utilitarian social welfare relation. They apply this criterion to a one-dimension neoclassical Ramsey model. In fact, they implicitely use a utilitarian undiscounted social welfare function, as in Gale [1967], Brock [1970a], Dana and Le Van [1990] and Dana and Le Van [1993].

Chichilnisky [1996] proposes two axioms that capture the idea of sustainable growth: no dictatorship of the present and no dictatorship of the future. A social welfare function is said to give a dictatorial role to the present if it disregards the utilities of all generations from some generation on. Conversely, a social welfare function gives a dictatorial role to the future if it is only sensitive to the utilities of the generations coming after some generation. She exhibits, for the set of uniformly bounded consumptions, a social welfare criterion satisfying, besides the Pareto and independence requirements, these two axioms for sustainability, under a fairly general set of assumptions. She also claims that this criterion is the only one to do so. Chichilnisky's criterion is a mixed criterion, adding a discounted utilitarian part (with possibly non constant discount rates), which gives a dictatorial role to the present, and a long term part, which gives a dictatorial role to the future.

Chichilnisky [1996] examines the ability of some of the other criteria to define sustainable preferences, in the sense that they satisfy the two axioms for sustainability in her general framework. She shows for example that the Ramsey's criterion fails because it is not a well defined real valued function on all l^{∞}[2], and cannot therefore define a complete order on l^{∞}. The overtaking criterion also fails because it is not a well defined function of l^{∞}, since it cannot rank a pair of utility streams of l^{∞} for which neither the first overtakes the second, nor the second overtakes the first. The maximin criterion fails because it does not satisfy the independence property.

The drawback of Chichilnisky's criterion is that it often does not allow to obtain explicit optimal growth paths with optimal control techniques. Chichilnisky [1996] does not apply her criterion to growth models but confines herself the axiomatic approach. Applications are presented in Chichilnisky [1997] and Heal [1998], in the framework of canonical optimal growth models with exhaustible or renewable resources. Using non-standard techniques, they obtain a solution in the case of exhaustible resources, but they show that no solution exists in the model with renewable resources. Figuières and Tidball [2006] focus on this inexistence result, and show that they can find in the model with renewable resources "near optimal" growth paths. Moreover, we observe that the optimal solution obtained with Chichilnisky's criterion, cannot in general be approxi-

[2] $l^{\infty} = \left\{ (y_g)_{g=1,\dots\infty} : \; y_g \in \mathbb{R}, \; \sup_g |y_g| < \infty \right\}.$

mated by a sequence of optimal solutions with finite horizon.

Observe that the criterion proposed by Chichilnisky does not satisfy Anonymity, and Basu and Mitra [2007] do not prove that their criterion satisfies non-dictatorship requirements.

Our aim is less general than Chichilnisky's, and also less than Basu and Mitra: we want to have a non-dictatorial, anonymous and paretian criterion for optimal growth models. Instead of l_+^∞ as set of utilities, we just consider the set of utilities of consumptions which are generated by a specific technology. We define an independent sustainable welfare criterion for an economy defined by a technology with decreasing returns for high levels of capital. That means our criterion is specific for this economy. Our results are therefore less general, but we can characterize optimal growth paths[3].

Our main result is to show that the undiscounted utilitarian criterion pioneered by Ramsey [1928] is an independent non-dictatorial welfare criterion for an economy which satisfies the assumptions listed in Section 2. Besides, Gale [1967], Dana and Le Van [1990] and Dana and Le Van [1993] require that the technology must be of decreasing returns. Here, we drop this assumption. We show that the turnpike result in Gale [1967] also holds with non-convex technologies. Another advantage of the use of this criterion is that any good programme, i.e. any programme for which the intertemporal utility is well defined, converges to a steady state corresponding to the Golden Rule. But our undiscounted social welfare function satisfies also the Anonymity and Pareto axioms. We then show that in a muti-dimension optimal growth model, the capital stocks which are optimal by using the undiscounted utilitarian criterion are maximal for the Basu-Mitra criterion. We extend their result obtained for the one-dimension case. To sum up, if we just consider the good programmes (as defined by Gale [1967]), we obtain a social welfare function which is not only convenient if one wants to solve an optimal growth problem but also sustainable, efficient and equitable.

The paper is organized as follows. In Section 2 we set the model, Section 3 introduces the good programmes and proves their turnpike property. Section 4 recalls the concept of non-dictatorship for the welfare criteria. Section 5 defines the independent and non-dictatorial criterion for our model. Applications are set in Section 6. In particular, we present an economy with a convex-concave production function (see Dechert and Nishimura [1983]). In Dechert and Nishimura [1983], the proof of existence of a poverty trap is given, but its precise value is not easy to compute. As a by product, we obtain here the explicit value of this poverty trap. We also present two growth models respec-

[3]Note however that Chichilnisky [1996], Basu and Mitra [2007] impose that utilities are in $[0,\ 1]^{\mathbb{N}}$. By doing so, they implicitely assume a bounded technology.

tively with exhaustible and renewable resources. Section 7 is devoted to some comments. In particular, we show there that we obtain a social welfare function on the set of good programmes and the optimal path is a maximal element for the Basu-Mitra criterion.

2 The set-up

We consider an intertemporal economy where the instantaneous utility of the representative consumer depends on x_t, the capital stock on hand at date t, and on x_{t+1}, the capital stock for date $t+1$. Given x_t, the set of feasible capital stocks for the next period $t+1$ is $\Gamma(x_t)$. We assume that at any period t the feasible capital stock on hand belongs to X, a subset of $\mathbb{R}^n_+$. More explicitly, we make the following assumptions[4].

Assumptions

H1: $X = \cup_{i=1}^{+\infty} K_j$, where $\{K_j\}$ is a sequence of increasing (i.e. $K_j \subseteq K_{j+1}, \ \forall j$) compact, convex sets of $\mathbb{R}^n_+$ with non-empty interior, and X contains 0.

H2: Γ is a continuous correspondence with non-empty images. It satisfies $\forall j, \Gamma(K_j) \subseteq K_j, \ \forall j$. Its graph is the set $\text{graph}\Gamma = \{(x, y) \in X \times X : y \in \Gamma(x)\}$.

H3: intgraphΓ, the interior of the graph of Γ, is non-empty.

H4: For any $x \in X$, $0 \in \Gamma(x)$.

We will denote by cographΓ the convex hull of graphΓ.

H5: The instantaneous utility function u : cograph$\Gamma \to \mathbb{R}$ is strictly concave and continuous. It is increasing with respect to the first variable and decreasing with respect to the second variable.

For the simplicity of our presentation we also assume the following.

H6: The function u is differentiable in intcographΓ.

H7: The set $I(X) = \{x \in X : (x, x) \in \text{ graph}\Gamma\}$ is compact.

Remark 1 *1. Dana and Le Van [1990] show that the following assumptions imply intgraph*$\Gamma \neq \emptyset$ ***(H3)**:.*

- Free disposal assumption: If $y \in \Gamma(x)$, $x' \geq x$, $y' \leq y$, $x' \in X$, $y' \in X$, *then* $y' \in \Gamma(x')$.

- Existence of expansible capital stocks: There exists $(x, y) \in$ *graph*Γ, *with* $y >> x$, *i.e.* $y_i > x_i$, $\forall i \in \{1, ..., n\}$.

2. Moreover, the Free disposal assumption implies $0 \in \Gamma(x)$ *for any* $x \in X$ *(**H4**).*

[4]This set-up is borrowed from Dana and Le Van [1990], but we do not assume neither that graphΓ is convex, i.e. we do not impose a technology with decreasing returns to scale, nor that X is compact.

3. If Γ *is continuous and satisfies the condition:*

There exists a number a such that $\{\|x\| \geq a, \; y \in \Gamma(x)\} \Rightarrow \|y\| < \|x\|$, *then* $I(X)$ *is compact.*

*4. An example where **H1–H2** and **H7** are satisfied. Let* $X = \mathbb{R}_+$ *and* $K_0 = [0, \; 1], .., K_n = [0, n+1], ...$ *Let* $\Gamma(x) = \{y \in \mathbb{R}_+ : y \in [0, f(x)]\}$, *where* f *is a continuous increasing function which satisfies* $f(0) = 0, \; f(1) = 1, \; f(x) < x, \; \forall x > 1$.

Obviously, when X *is compact then **H1–H2** and **H7** are satisfied.*

A sequence $\mathbf{x}$ is feasible from $x_0 \in X$ if, $\forall t \geq 0$, $x_{t+1} \in \Gamma(x_t)$. We denote by $\Pi(x_0)$ the set of feasible sequences from x_0. Following Gale [1967], a *programme* from x_0 is a feasible sequence from x_0. The set of programmes is denoted by Π, i.e. $\Pi = \cup_{x \in X} \Pi(x)$.

Lemma 1 *Assume **H1–H2**. Then* $\Pi(x_0)$ *is compact for the product topology.*

Proof: See e.g. Le Van and Dana [2003], chapter 4. ■

3 Good Programmes

Definition 1 *A stationary point* $(\overline{x}, \overline{x})$ *satisfies*

$$\overline{u} = u(\overline{x}, \overline{x}) = \max \{u(x, x) : (x, x) \in graph\Gamma\}.$$

Proposition 1 *There exists a stationary point* $(\overline{x}, \overline{x})$ *which satisfies*
(i) $\overline{u} = u(\overline{x}, \overline{x}) = \max \{u(x, x) : (x, x) \in graph\Gamma\}$;
(ii) if $(\overline{x}, \overline{x}) \in intgraph\Gamma$, *then*

$$u(x, y) + u_1(\overline{x}, \overline{x})y - u_1(\overline{x}, \overline{x})x \leq u(\overline{x}, \overline{x}), \; \forall (x, y) \in \; graph\Gamma.$$

Proof: (i) Since $I(X)$ is compact, the proof of the existence of $(\overline{x}, \overline{x})$ is obviuous.
(ii) If $(\overline{x}, \overline{x})$ is interior then $u_1(\overline{x}, \overline{x}) + u_2(\overline{x}, \overline{x}) = 0$. Since u is concave, for any $(x, y) \in \text{graph}\Gamma$, we have

$$u(\overline{x}, \overline{x}) - u(x, y) \geq u_1(\overline{x}, \overline{x})(\overline{x} - x) + u_2(\overline{x}, \overline{x})(\overline{x} - y) = u_1(\overline{x}, \overline{x})y - u_1(\overline{x}, \overline{x})x$$

■

We now add another assumption.

H8: $(\overline{x}, \overline{x}) \in \text{intgraph}\Gamma$.

Assumption **H8**.is not very restrictive, since in the one dimension Ramsey model it implies that consumption is strictly positive at the Golden Rule. Indeed, if $\Gamma(x) = \{y : 0 \le y \le f(x)\}$ where f is the production function, then $\overline{x}$ satisfies $\overline{u} = \max_x u(f(x) - x)$.

Let $\bar{p} = u_1(\overline{x}, \overline{x})$.

Proposition 2 *Under **H1-H8**, $(\overline{x}, \overline{x})$ is the unique solution to problem*

$$\max \{u(x, y) + \bar{p}y - \bar{p}x : (x, y) \in \ \mathit{graph}\Gamma\}.$$

Proof: Let $(\hat{x}, \hat{y})$ be another solution. Then

$$u(\hat{x}, \hat{y}) + \bar{p}\hat{y} - \bar{p}\hat{x} = u(\bar{x}, \bar{x}). \tag{1}$$

Since $(\overline{x}, \overline{x}) \in \text{intgraph}\Gamma$, for $\lambda \in (0, 1)$, close enough to 1, we have

$$(\lambda\overline{x} + (1 - \lambda)\hat{x}, \lambda\overline{x} + (1 - \lambda)\hat{y}) \in \ \text{graph}\Gamma.$$

Denote $x_\lambda = \lambda\overline{x} + (1 - \lambda)\hat{x}$, $y_\lambda = \lambda\overline{x} + (1 - \lambda)\hat{y}$. Then

$$\lambda u(\overline{x}, \overline{x}) + (1 - \lambda)u(\hat{x}, \hat{y}) + \bar{p}y_\lambda - \bar{p}x_\lambda < u(x_\lambda, y_\lambda) + \bar{p}y_\lambda - \bar{p}x_\lambda \le u(\overline{x}, \overline{x}),$$

and then

$$\lambda u(\overline{x}, \overline{x}) + (1 - \lambda)\left[u(\hat{x}, \hat{y}) + \bar{p}\hat{y} - \bar{p}\hat{x}\right] < u(\overline{x}, \overline{x})$$

or equivalently

$$u(\hat{x}, \hat{y}) + \bar{p}\hat{y} - \bar{p}\hat{x} < u(\overline{x}, \overline{x})$$

in contradiction with (1). ■

Theorem 1 *Assume **H1-H8**. For any programme* $\mathbf{x}$*, either*
(i) $\lim_{T\to+\infty} \sum_{t=0}^{T} [u(x_t, x_{t+1}) - \bar{u}]$ *exists in* $\mathbb{R}$ *and* $x_t \to \overline{x}$*,*
or (ii) $\lim_{T\to+\infty} \sum_{t=0}^{T} [u(x_t, x_{t+1}) - \bar{u}] = -\infty$.

Proof: Theorem 1 is identical to Proposition 1.3.1 and Corollary 1.3.2. in Le Van and Dana [2003]. But we have to modify the proof since graphΓ is not assumed to be convex. Consider $\lim_{T\to+T} \sum_{t=0}^{T} [u(x_t, x_{t+1}) + \bar{p}x_{t+1} - \bar{p}x_t - \bar{u}]$ which exists in $\mathbb{R}_- \cup \{-\infty\}$ since $\forall t,\ u(x_t, x_{t+1}) + \bar{p}x_{t+1} - \bar{p}x_t - \bar{u} \le 0$. We have

$$\sum_{t=0}^{T} [u(x_t, x_{t+1}) + \bar{p}x_{t+1} - \bar{p}x_t - \bar{u}] = \sum_{t=0}^{T} [u(x_t, x_{t+1}) - \bar{u}] - \bar{p}(x_0 - x_{T+1}) \tag{2}$$

If $\lim_{T\to+\infty} \sum_{t=0}^{T} [u(x_t, x_{t+1}) + \bar{p}x_{t+1} - \bar{p}x_t - \bar{u}] \in \mathbb{R}_-$ then $u(x_t, x_{t+1}) + \bar{p}x_{t+1} - \bar{p}x_t \to \bar{u}$. There exists j such that $x_0 \in K_j$. That implies $x_t \in K_j,\ \forall t$. Let $(\hat{x}, \hat{y})$

be a cluster point of $\{(x_t, x_{t+1})\}$. Then $u(\hat{x}, \hat{y}) + \bar{p}\hat{y} - \bar{p}\hat{x} = \bar{u}$. From Proposition 2, $(\hat{x}, \hat{y}) = (\overline{x}, \overline{x})$. Therefore the sequence (x_t) converges to $\overline{x}$. From (2), $\sum_{t=0}^{\infty} [u(x_t, x_{t+1}) - \bar{u}] \in \mathbb{R}$.

Relation (2) implies

$$\sum_{t=0}^{T} [u(x_t, x_{t+1}) - \bar{u}] \leq \sum_{t=0}^{T} [u(x_t, x_{t+1}) + \bar{p}x_{t+1} - \bar{p}x_t - \bar{u}] + |\bar{p}.x_0| + |\bar{p}.x_{T+1}|.$$

Hence, if

$$\lim_{T \to +\infty} \sum_{t=0}^{T} [u(x_t, x_{t+1}) + \bar{p}x_{t+1} - \bar{p}x_t - \bar{u}] = -\infty,$$

then $\lim_{T \to +\infty} \sum_{t=0}^{\infty} [u(x_t, x_{t+1}) - \bar{u}] = -\infty$, since the sequence (x_{T+1}) is bounded.
■

Definition 2 *A programme* $\mathbf{x} \in \Pi(x_0)$ *is good if* $\lim_{T \to \infty} \sum_{t=0}^{T} [u(x_t, x_{t+1}) - u(\overline{x}, \overline{x})]$ *exists in* $\mathbb{R}$.

Definition 3 *The set of the good programmes* $\mathbf{x} \in \Pi(x_0)$ *is denoted* $G(x_0)$.

¿From Theorem 1 one gets:

Corollary 1 *Assume **H1-H8**. If* $\mathbf{x} \in G(x_0)$*, then* $x_t \to \overline{x}$.

The following lemma, which seems to be just technical, is actually crucial to prove that the criterion we propose is non-dictatorial.

Lemma 2 *For any* $x_0 \in X$*, the set of good programmes from* x_0 *is open (for the product topology) in* $\Pi(x_0)$.

Proof: If $G(x_0)$ is empty, the proof is over. So, assume $G(x_0) \neq \emptyset$. Let $\mathbf{x} \in G(x_0)$. We first claim that there exists T such that $(x_T, x_{T+1}) \in \text{intgraph}\Gamma$, and $(x_{T+1}, x_{T+2}) \in \text{intgraph}\Gamma$.
Indeed, there exists T_1 such that $(x_{T_1}, x_{T_1+1}) \in \text{intgraph}\Gamma$. If not, $\forall t$, $(x_t, x_{t+1}) \in \partial\text{graph}\Gamma$ and, by taking the limit, $(\overline{x}, \overline{x}) \in \partial\text{graph}\Gamma$ since $\partial\text{graph}\Gamma$ is closed: contradiction with assumption **H8**.
Since the sequence $(x_t)_{t \geq T_1+1}$ belongs to $G(x_{T_1})$, there exists an infinite sequence $\{(x_{T_n}, x_{T_n+1})\} \subset \text{int graph}\Gamma$. Consider the sequence $\{(x_{T_n+1}, x_{T_n+2})\}$. If for any n, $(x_{T_n+1}, x_{T_n+2}) \in \partial\text{graph}\Gamma$ we obtain again a contradiction $(\overline{x}, \overline{x}) \in \partial\text{graph}\Gamma$. Hence there exists n such that $(x_{T_n+1}, x_{T_n+2}) \in \text{int graph}\Gamma$. The claim is proved.
Let T satisfy $(x_T, x_{T+1}) \in \text{int graph}\Gamma$, $(x_{T+1}, x_{T+2}) \in \text{int graph}\Gamma$. There exists an open ball $B(x_{T+1}, \varepsilon)$ such that for any $y \in B(x_{T+1}, \varepsilon)$ we have $(x_T, y) \in$ int

graphΓ, $(y, x_{T+1}) \in$ int graphΓ. The sequence obtained from (x_t) by replacing x_{T+1} by y is in $G(x_0)$. That proves the openness of $G(x_0)$ for the product topology. ■

4 Non-Dictatorial Criteria

We now define, following Chichilnisky [1996], the concepts of non-dictatorship of the present and non-dictatorship of the future.

Let $l_+^\infty = \{(a_t)_{t=0,\dots,+\infty} : a_t \geq 0,\ \forall t,\ \sup_t a_t < +\infty\}$. A criterion W is an increasing function from l_+^∞ into $\mathbb{R}$.

A criterion W exhibits **No Dictatorship of the Present** if for any $\mathbf{a} \in l_+^\infty$, $\mathbf{b} \in l_+^\infty$ which satisfy $W(\mathbf{a}) > W(\mathbf{b})$, then, for any N, there exist $k \geq N$, $(c_t)_{t \geq k+1}$, $(d_t)_{t \geq k+1}$ such that

$$W(a_0, ..., a_k, c_{k+1}, ..., c_{k+t}, ...) \leq W(b_0, ..., b_k, d_{k+1}, ..., d_{k+t}, ...).$$

A criterion W exhibits **No Dictatorship of the Future** if for any $\mathbf{a} \in l_+^\infty$, $\mathbf{b} \in l_+^\infty$ which satisfy $W(\mathbf{a}) > W(\mathbf{b})$, then, for any N, there exist $k \geq N$, $(c_t)_{t=0,\dots,k}$, $(d_t)_{t=0,..,k}$ such that

$$W(c_0, ..., c_k, a_{k+1}, ..., a_{k+t}, ...) \leq W(d_0, ..., d_k, b_{k+1}, ..., b_{k+t}, ...).$$

5 A non-dictatorial criterion for optimal growth models

We consider the economy set in Section 2. Let us consider the problem

$$\max \lim_{T \to +\infty} \sum_{t=0}^{T} [u(x_t, x_{t+1}) - u(\overline{x}, \overline{x})]$$

under the constraints

$$\forall t,\ x_{t+1} \in \Gamma(x_t),\ x_0 \text{ is given.}$$

Consider $\mathbf{x} \in \Pi(x_0)$ and the function $W : \Pi \to]-\infty, -\infty[$ defined by

$$W(\mathbf{x}) = \lim_{T \to \infty} \sum_{t=0}^{T} [u(x_t, x_{t+1}) - u(\overline{x}, \overline{x})].$$

Proposition 3 *Assume **H1–H8**. Then W is upper semi-continuous for the product topology.*

Proof: See Dana and Le Van [1990]. ■

Definition 4 ***No dictatorship of the present***

Assume that $\mathbf{x}$ *and* $\mathbf{x}'$ *are two good programmes which satisfy* $W(\mathbf{x}) > W(\mathbf{x}')$. *Then* $\forall N$, $\exists k \geq N$, $\exists \mathbf{z} \in G(x_{k+1})$, $\exists \mathbf{z}' \in G(x'_{k+1})$, *satisfying*

$$\begin{aligned} & \sum_{t=0}^{k} [u(x_t, x_{t+1}) - \overline{u}] + [u(x_{k+1}, z_{k+2}) - \overline{u}] + \sum_{t=k+2}^{\infty} [u(z_t, z_{t+1}) - \overline{u}] \\ \leq \ & \sum_{t=0}^{k} \left[u(x'_t, x'_{t+1}) - \overline{u}\right] + \left[u(x'_{k+1}, z'_{k+2}) - \overline{u}\right] + \sum_{t=k+2}^{\infty} \left[u(z'_t, z'_{t+1}) - \overline{u}\right]. \end{aligned}$$

Theorem 2 *Assume* ***H1–H8***. *W exhibits no dictatorship of the present.*

Proof: Assume $W(\mathbf{x}) > W(\mathbf{x}') > -\infty$ with $\mathbf{x} \in G(x_0)$, $\mathbf{x}' \in G(x'_0)$. Let N be given. Then the sequence $(x_t)_{t \geq N+2} \in G(x_{N+1})$. Observe that the set of bad programmes is not empty, since for any x_0, the programme $(x_0, 0, ..., 0, ...)$ is bad. That means $G(x_0)$ differs from $\Pi(x_0)$. In this case, let $(z_t)_{t \geq N+2} \in \partial G(x_{N+1})$. ¿From Lemma 1, it is not a good programme. There exists an infinite sequence $((z_t^n)_{t \geq N+2})_n$ of programmes in $G(x_{N+1})$ which converges to $(z_t)_{t \geq N+2}$ for the product topology. Since W is upper semi-continuous, we have

$$\begin{aligned} & \overline{\lim} \left[\sum_{t=0}^{N} [u(x_t, x_{t+1}) - \overline{u}] + \left[u(x_{N+1}, z^n_{N+2}) - \overline{u}\right] + \sum_{t=N+2}^{\infty} \left[u(z^n_t, z^n_{t+1}) - \overline{u}\right] \right] \\ \leq \ & \left[\sum_{t=0}^{N} [u(x_t, x_{t+1}) - \overline{u}] + [u(x_{N+1}, z_{N+2}) - \overline{u}] + \sum_{t=N+2}^{\infty} [u(z_t, z_{t+1}) - \overline{u}] \right] = -\infty. \end{aligned}$$

Hence, for N large enough,

$$\begin{aligned} & \left[\sum_{t=0}^{N} [u(x_t, x_{t+1}) - \overline{u}] + \left[u(x_{N+1}, z^n_{N+2}) - \overline{u}\right] + \sum_{t=N+2}^{\infty} \left[u(z^n_t, z^n_{t+1}) - \overline{u}\right] \right] \\ \leq \ & \sum_{t=0}^{\infty} \left[u(x'_t, x'_{t+1}) - \overline{u}\right]. \end{aligned}$$

■

Definition 5 ***No dictatorship of the future***

Let $W(\mathbf{x}) > W(\mathbf{x}') > -\infty$. *Then* $\forall N$, $\exists k \geq N$, $\exists (z_0, ..., z_{k+1})$, $\exists (z'_0, ..., z'_{k+1})$, *satisfying*

$$\begin{aligned} & \sum_{t=0}^{k} [u(z_t, z_{t+1}) - \overline{u}] + [u(z_{k+1}, x_{k+2}) - \overline{u}] + \sum_{t=k+2}^{\infty} [u(x_t, x_{t+1}) - \overline{u}] \\ \leq \ & \sum_{t=0}^{k} \left[u(z'_t, z'_{t+1}) - \overline{u}\right] + \left[u(z'_{k+1}, x'_{k+2}) - \overline{u}\right] + \sum_{t=k+2}^{\infty} \left[u(x'_t, x'_{t+1}) - \overline{u}\right]. \end{aligned}$$

Theorem 3 *Assume **H1–H8**. W exhibits no dictatorship of the future.*

Proof: Assume $W(\mathbf{x}) > W(\mathbf{x}')$ with $\mathbf{x} \in G(x_0)$, $\mathbf{x}' \in G(x_0')$. Let N be given. Since $(\overline{x}, \overline{x}) \in \text{int graph}\Gamma$, there exists an open ball $B((\overline{x}, \overline{x}), \rho) \in \text{int graph}\Gamma$. Take $(\overline{x}', \overline{x}') \in B((\overline{x}, \overline{x}), \frac{\rho}{2})$, with $(\overline{x}', \overline{x}') \neq (\overline{x}, \overline{x})$. Since $(x_t) \to \overline{x}$, $\exists k \geq N$ such that $\forall t \geq k$, $(x_t, x_{t+1}) \in B((\overline{x}, \overline{x}), \frac{\rho}{2})$ and $(\overline{x}', x_t) \in \text{int graph}\Gamma$. Let $\varepsilon > 0$. Then for any k large enough $\left|\sum_{t=k+1}^{\infty} [u(x_t, x_{t+1}) - \overline{u}]\right| \leq \varepsilon$. Observe that $u(\overline{x}', \overline{x}') - \overline{u} < 0$ by assumption **H7** and if $\xi_t = \overline{x}'$ $\forall t = 0, ..., \infty$, then $\sum_{t=0}^{\infty} \left[u(\xi_t, \xi_{t+1}) - \overline{u}\right] = -\infty$. Let $z_t = \overline{x}'$ $\forall t = 0, ..., k$. Then

$$\begin{aligned} & \sum_{t=0}^{k-1} [u(z_t, z_{t+1}) - \overline{u}] + [u(z_k, x_{t+1}) - \overline{u}] + \sum_{t=k+1}^{\infty} [u(x_t, x_{t+1}) - \overline{u}] \\ \leq & \sum_{t=0}^{k-1} [u(z_t, z_{t+1}) - \overline{u}] + \max_{(x,y) \in \text{ graph}\Gamma} [u(x, y) - \overline{u}] + \varepsilon, \end{aligned}$$

and

$$\lim_{k \to \infty} \left[\sum_{t=0}^{k-1} [u(z_t, z_{t+1}) - \overline{u}] + [u(z_k, x_{t+1}) - \overline{u}] + \sum_{t=k+1}^{\infty} [u(x_t, x_{t+1}) - \overline{u}] \right] = -\infty.$$

Therefore, for k large enough,

$$\begin{aligned} & \sum_{t=0}^{k-1} [u(z_t, z_{t+1}) - \overline{u}] + [u(z_k, x_{t+1}) - \overline{u}] + \sum_{t=k+1}^{\infty} [u(x_t, x_{t+1}) - \overline{u}] \\ \leq & \sum_{t=0}^{\infty} \left[u(x_t', {x'}_{t+1}) - \overline{u}\right]. \end{aligned}$$

■

Theorem 4 *Assume **H1–H8**. If $G(x_0 \neq \emptyset$, then there exists an optimal path. It is unique if graphΓ is convex.*

Proof: ¿From proposition 3 and lemma 1, for the product topology, W is usc and $\Pi(x_0)$ is compact. Therefore, an optimal path exists, since the problem is to maximize $W(\mathbf{x})$ with $\mathbf{x} \in \Pi(x_0)$. This optimal path is unique when graphΓ is convex since u is assumed strictly concave. ■

6 Applications

6.1 Convex technology

Assume graphΓ is convex. Now, let $X = \mathbb{R}_+$, $K_0 = [0,\ 1]$, $K_j = [0,\ j+1]$, $j \geq 1$, $\Gamma(x) = [0,\ f(x)]$ where f is strictly concave, increasing, differentiable with

$f'(0) > 1$, $f(0) = 0$, $f(1) = 1$, and $u(x, y) = v(f(x) - y)$, where v is real-valued, defined on $\mathbb{R}_+$, increasing, strictly concave, differentiable. It is easy to check **H1-H7**. Observe that the stationary point $\overline{x}$ is defined by $f'(\overline{x}) = 1$. It is therefore unique. Obviously, $0 < \overline{x} < f(\overline{x}) < 1$. In other words, **H8** is satisfied. We claim that $G(x_0) \neq 0$, $\forall x_0 > 0$. Indeed, since $f^t(x_0) \to 1$ when $t \to +\infty$, there exists T such that $f^T(x_0) \leq f(\overline{x})$ and $f^{T+1}(x_0) > f(\overline{x})$. The sequence $(x_0, f(x_0), ..., f^T(x_0), \overline{x}, \overline{x}, ..., \overline{x}, ...) \in G(x_0)$.

6.2 Economy with a convex-concave production function

Let $X = \mathbb{R}_+$, $K_0 = [0,\ 1]$, $K_j = [0,\ j+1]$, $j \geq 1$, $\Gamma(x) = [0,\ f(x)]$. Here we suppose that f is increasing, continuously differentiable, $f(0) = 0$, $f(1) = 1$, strictly convex between 0 and $x_I < 1$ and strictly concave in $[x_I,\ +\infty[$. The function u is as before $u(x, y) = v(f(x) - y)$, where v is real-valued, defined on $\mathbb{R}_+$, increasing, strictly concave, differentiable.

Let $a = \max_{x \in X} \left\{ \frac{f(x)}{x} \right\}$ and $x_a \in X$ satisfy $a = \left\{ \frac{f(x_a)}{x_a} \right\}$. Let F be defined as follows: $F(x) = ax$, $\forall x \leq x_a$; $F(x) = f(x)$, $x \geq x_a$. Then F is concave and $\{(x, y) \in X \times X : y \leq F(x)\}$ is cographΓ.

When $f'(0) > 1$, there exists a unique $\overline{x} \in (x_I,\ 1)$ such that $f'(\overline{x}) = 1$. One can check that $0 < \overline{x} < f(\overline{x}) < 1$, i.e. $(\overline{x}, \overline{x}) \in$ int graphΓ. Assumptions **H1–H8** are satisfied. Again $G(0) = \emptyset$ and $G(x_0) \neq \emptyset$ if $x_0 > 0$.

Now assume $f'(0) < 1 < \frac{f(x_a)}{x_a}$ and $f'(x_I) > 1$ (see figure 1). There are two points $\underline{x}$, $\overline{x}$ such that $f'(\underline{x}) = f'(\overline{x}) = 1$. But $f(\underline{x}) - \underline{x} < 0$, i.e. $(\underline{x}, \underline{x}) \notin$ graphΓ. We have a unique stationary point $\overline{x}$. It satisfies $0 < \overline{x} < f(\overline{x}) < 1$, i.e. $(\overline{x}, \overline{x}) \in$ int graphΓ. Again, **H1–H8** are satisfied.

Let $\tilde{x}$ satisfy $f(\tilde{x}) = \tilde{x}$. then $\tilde{x} \in (0,\ 1)$. Moreover, $x < \tilde{x} \Rightarrow f(x) < x$ and $1 > x > \tilde{x} \Rightarrow f(x) > x$. That implies

(i) any feasible path from $x_0 < \tilde{x}$ will converge to 0. Hence, $G(x_0) = \emptyset$ for any $x_0 \leq \tilde{x}$.

(ii) for any $x_0 > \tilde{x}$, $f^t(x_0) \to 1$. Hence, one can find T such that $f^T(x_0) \leq \overline{x}$ and $f^{T+1}(x_0) > f(\overline{x})$. The sequence $(x_0, f(x_0), ..., f^T(x_0), \overline{x}, \overline{x}, ..., \overline{x}, ...) \in G(x_0)$. Thus, $G(x_0) \neq \emptyset$ for any $x_0 > \tilde{x}$.

One concludes that $\tilde{x}$ is the poverty trap.

6.3 Growth and exhaustible resources

We consider the model in Le Van, Schubert and Nguyen [2007]. The country possesses a stock of a non-renewable natural resource S_0. This resource is extracted at a rate R_t, and then sold abroad at a price P_t, in terms of the numeraire, which is the domestic single consumption good. The inverse demand function for the resource is $P(R_t)$. The revenue from the sale of the natural resource, $\phi(R_t) = P(R_t)R_t$, is used to buy a foreign good, which is

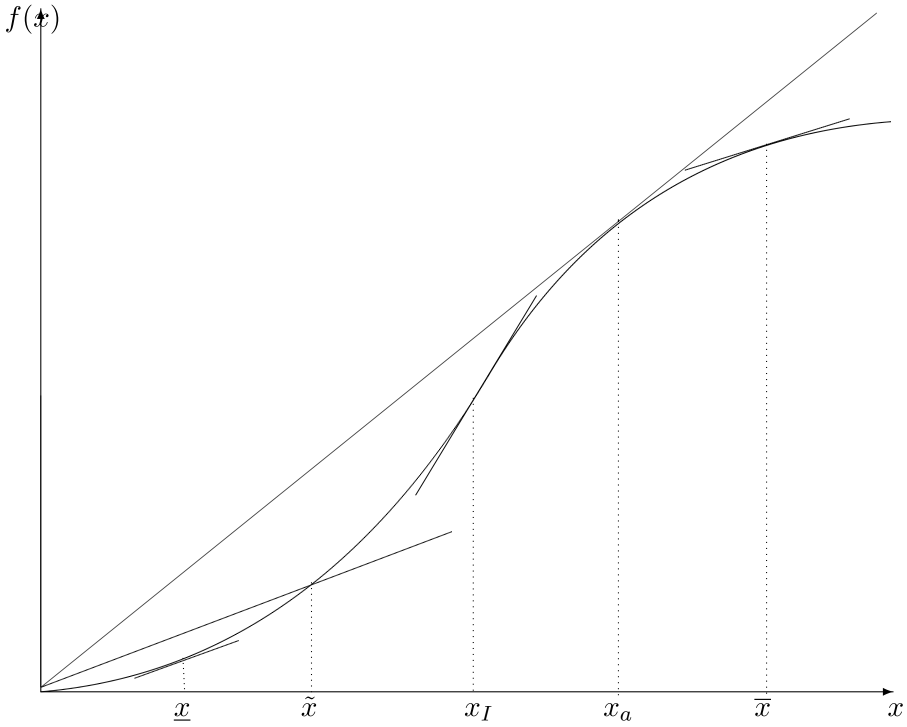

Figure 1: The convex-concave technology

supposed to be a perfect substitute of the domestic good, used for consumption and capital accumulation. The domestic production function is $F(k_t)$, where k_t is physical capital. The depreciation rate is δ. We define the function $f(k_t) = F(k_t) + (1-\delta)k_t$, and we name it for simplicity the technology. The constraints of the economy are:

$$\begin{aligned}
\forall t, c_t &\geq 0, k_t \geq 0, R_t \geq 0, \\
c_t + k_{t+1} &\leq f(k_t) + \phi(R_t), \\
\sum_{t=0}^{+\infty} R_t &\leq S_0, \\
S_0 &> 0,\ k_0 \geq 0 \text{ are given,}
\end{aligned}$$

where c_t is consumption. We assume that f is strictly concave, $f(0) = 0,\ f'(+\infty) < 1,\ f'(0) > 1$. Let $\tilde{k}$ satisfy $\tilde{k} = f(\tilde{k}) + \phi(S_0)$. Such a $\tilde{k}$ is unique. Take $X = [0,\ \tilde{k}] \times [0,\ S_0]$. Let $x = (k, R)$. Define $\Gamma(x) = \{(y, R) :\ 0 \leq y \leq f(k) + \phi(R)\}$. One can check that Γ is continuous and maps X into X. Observe that any feasible sequence $\{R_t\}$ will converge to zero. We also assume $\phi'(0) < +\infty$. The representative consumer has an instaneous utility function u which is increasing, strictly concave, satisfies $u(0) = 0$. We show that the methodology of Sections 2–4 may be used for the present model.

Let $\bar{k}$ satisfy

$$\begin{aligned} u(f(\bar{k})-\bar{k}) &= \max\{u(f(k)-k):\ 0\le k\le f(k)\} \\ &= \max\{u(f(k)-k):\ 0\le k\le f(k)+\phi(S_0)\} \end{aligned}$$

Such a $\bar{k}$ is unique and satisfies $0<\bar{k}<f(\bar{k})$ and $f'(\bar{k})=1$. Let $\bar{p}=u'(f(\bar{k})-\bar{k})$. We let to the reader to prove the following claim (see the proofs of Propositions 1, 2).

Claim 1

$(\bar{k},\bar{k},0)$ is the unique solution to problem

$$\max\left\{u(f(k)-y+\phi(R))+\bar{p}y-\bar{p}k-\bar{p}\phi'(0)R : ((k,R),(y,R))\in \text{ graph}\Gamma\right\}.$$

Let $\bar{u}=u(f(\bar{k})-\bar{k})$.

Claim 2 For any program $(\mathbf{k},\mathbf{R})$, either

(i) $\lim_{T\to+\infty}\sum_{t=0}^{T}[u(f(k_t)-k_{t+1}+\phi(R_t))-\bar{u}]$ exists in $\mathbb{R}$ and $k_t\to\bar{k}$, $R_t\to 0$,

or (ii) $\lim_{T\to+\infty}\sum_{t=0}^{T}[u(f(k_t)-k_{t+1}+\phi(R_t))-\bar{u}]=-\infty$

Proof of Claim 2

(i) and (ii): Observe that

$$\begin{aligned} &\sum_{t=0}^{T}\left[u(f(k_t)-k_{t+1}+\phi(R_t))+\bar{p}k_{t+1}-\bar{p}k_t-\bar{p}\phi'(0)R_t-\bar{u}\right] \\ = &\sum_{t=0}^{T}[u(f(k_t)-k_{t+1}+\phi(R_t))-\bar{u}]+\bar{p}(k_{T+1}-k_0)-\bar{p}\phi'(0)\sum_{t=0}^{T}R_t. \end{aligned}$$

Since $\sum_{t=0}^{T}R_t$ converges, use Claim 1 and the proof of Theorem 1 to conclude.

A programme $\{(\mathbf{k},\mathbf{R})\}$ is *good* if $\sum_{t=0}^{+\infty}[u(f(k_t)-k_{t+1}+\phi(R_t))-\bar{u}]\in\mathbb{R}$.

Let

$$Z(k_0,S_0)=\{(k,y):\exists R\in[0,\ S_0],\ 0\le y\le f(k)+\phi(R)\}.$$

Obviously, $(\bar{k},\bar{k})\in \text{int}Z(k_0,S_0)$. Let

$$\Psi(k_0,S_0)=\{\mathbf{k}:\exists\{R_t\},\ \forall t,\ R_t\in[0,\ S_0],\ \{(\mathbf{k},\mathbf{R})\}\text{ is a good programme}\}.$$

Claim 3 $\Psi(k_0,S_0)$ is open for the product topology.

Proof of Claim 3

If $\Psi(k_0,S_0)$ is empty, we are done. So, let $\mathbf{k}\in\Psi(k_0,S_0)$. Use the same kind of proof as in Lemma 2 and the openness of $\Psi(k_0,S_0)$.

Now define

$$W(\mathbf{k}, \mathbf{R}) = \lim_{T \to +\infty} \sum_{t=0}^{T} \left[u(f(k_t) - k_{t+1} + \phi(R_t)) - \bar{u} \right].$$

W is upper semi-continuous and non-dictatorial. The optimal solution $(\mathbf{k}^*, \mathbf{R}^*)$ to

$$\max_{(\mathbf{k},\mathbf{R})} W(\mathbf{k}, \mathbf{R})$$

under the constraints

$$\begin{aligned} \forall t,\ 0 \le k_{t+1} &\le f(k_t) + \phi(R_t),\ k_0 \text{ is given} \\ \sum_{t=0}^{+\infty} R_t &\le S_0, \end{aligned}$$

will converges to $(\bar{k}, 0)$.

6.4 Growth and renewable resources

We study the canonical model of growth with a renewable resource. The economy possesses a stock of a renewable natural resource S_0. This resource is extracted at a rate R_t. The domestic production function is $F(k_t, R_t)$. k_t is the stock of physical capital. The depreciation rate is δ. The natural growth of the renewable resource is $S_{t+1} - S_t = H(S_t)$. The function H is strictly concave, differentiable, satisfies $H(0) = 0$, $H(\hat{S}) = 0$, with $\hat{S} > 0$. There is a representative consumer with a utility function u depending on her consumption c. The technological constraints for each period t are as follows:

$$\begin{aligned} c_t + k_{t+1} &\le F(k_t, R_t) + k_t(1 - \delta) \\ S_{t+1} &\le S_t + H(S_t) - R_t. \end{aligned}$$

Let $\psi(S) = S + H(S)$. The function ψ is strictly concave, $\psi(0) = 0,\ \psi'(+\infty) < 1,$ since $H'(+\infty) < 0$. We define the function $f(k_t, R_t) = F(k_t, R_t) + (1 - \delta)k_t$, and we name it for simplicity the technology. We assume that f is strictly concave, $f(0, R) = f(k, 0) = 0,\ f_1(+\infty, \psi(\hat{S})) < 1,\ f_1(0, \psi(\hat{S})) > 1$.

The constraints become

$$\begin{aligned} 0 \le k_{t+1} &\le f(k_t, R_t) \\ 0 \le S_{t+1} &\le \psi(S_t) - R_t. \end{aligned}$$

Let $\hat{k}$ satisfy $\hat{k} = f(\hat{k}, \psi(\hat{S}))$. Let $X = [0,\ \hat{k}] \times [0,\ \hat{S}]$ and for $x = (k, S)$ define

$$\Gamma(x) = \left\{ x' = (k', S') : 0 \le k' \le f(k, \psi(S) - S'),\ 0 \le S' \le \psi(S) \right\}$$

Γ is continuous and maps X into X. Its graph is convex. There exists a unique $\bar{x} = (\bar{k}, \bar{S})$ such that $(\bar{x}, \bar{x})$ is in the interior of graphΓ and solves

$$\max \{u(f(k, \psi(S) - S) - k) : ((k, S), (k, S)) \in \text{ graph}\Gamma\}.$$

Namely, $f_1(\bar{k}, \psi(\bar{S}) - \bar{S}) = 1$, $\psi'(\bar{S}) = 1$. This model satisfies assumptions **H1–H7**. Hence, one can define

$$W(\mathbf{k}, \mathbf{S}) = \lim_{T \to +\infty} \sum_{t=0}^{T} [u(f(k_t, \psi(S_t) - S_{t+1}) - k_{t+1}) - \bar{u}]$$

where $\overline{u} = u\left(f(\overline{k}, \psi(\overline{S}) - \overline{S}) - \overline{k}\right)$. The optimal solution to

$$\max_{(\mathbf{k}, \mathbf{S})} W(\mathbf{k}, \mathbf{S})$$

under the constraints

$$\begin{aligned} \forall t,\ 0 \leq k_{t+1} &\leq f(k_t, \psi(S_t) - S_{t+1}), \\ 0 \leq S_{t+1} &\leq \psi(S_t), \quad (k_0, S_0) \text{ are given} \end{aligned}$$

will converges to $(\bar{k}, \bar{S})$ if there exists a good programme from (k_0, S_0).

7 Comments

1. Since our criterion W is a sum of utilities, it is independent in the sense of Chichilnisky [1996] (p. 246), that is the marginal rate of substitution between the utilities of two periods t_1 and t_2 depends only on the numbers t_1 and t_2. In particular, a criterion satisfying the independence property is represented by a linear function of the utility streams. In this perspective, the Rawls criterion, which can be shown to be non-dictatorial, is not independent. Observe that our criterion is also Paretian and anonymous.

2. As mentioned by Chichilnisky, in the discounted utilitarian criterion, the present is dictatorial while the future is not if we impose as in Chichinisky [1996] that $\sup_t a_t \leq 1$, where $a_t \geq 0$, $\forall t$.

Indeed, assume

$$\sum_{t=0}^{+\infty} \beta^t a_t > \sum_{t=0}^{+\infty} \beta^t a'_t$$

and

$$\delta = \sum_{t=0}^{+\infty} \beta^t a_t - \sum_{t=0}^{+\infty} \beta^t a'_t,$$

where $\beta < 1$ is the discount factor. Then there exists N, such that for any $(c_t)_t : 0 \leq c_t \leq 1$, any $(c'_t)_t : 0 \leq c'_t \leq 1$, we always have

$$\forall n > N, \sum_{t=n+1}^{+\infty} \beta^t c_t + \sum_{t=n+1}^{+\infty} \beta^t c'_t < \frac{\delta}{2}$$

and hence

$$\sum_{t=0}^{n} \beta^t a_t + \sum_{t=n+1}^{+\infty} \beta^t c_t > \sum_{t=0}^{n} \beta^t a'_t + \sum_{t=n+1}^{+\infty} \beta^t c'_t.$$

We have shown that the present is dictatorial.

Now, let

$$\sum_{t=0}^{+\infty} \beta^t a_t > \sum_{t=0}^{+\infty} \beta^t a'_t.$$

Then there exists N, such that for any $n > N$, $\sum_{t=n+1}^{+\infty} \beta^t a_t < \sum_{t=0}^{+\infty} \beta^t a'_t$. Take $b_t = 0, \quad \text{for } t = 0, ..., n$. Then

$$\sum_{t=0}^{n} \beta^t b_t + \sum_{t=n+1}^{+\infty} \beta^t a_t < \sum_{t=0}^{+\infty} \beta^t a'_t.$$

In other words, the future is not dictatorial.

But the condition $\sup_t a_t \leq 1$ is actually crucial to prove that the present is dictatorial in the discounted utilitarian criterion. To see that, drop the assumption $\sup_t a_t \leq 1$ and use l^∞_+. Again, let

$$\sum_{t=0}^{+\infty} \beta^t a_t > \sum_{t=0}^{+\infty} \beta^t a'_t.$$

Let N be given. Take $c_t = 0, \ \forall t \geq N+1$, and $c'_t = \gamma, t \geq N+1$, with $\gamma \sum_{t=N+1}^{+\infty} > \sum_{t=0}^{+\infty} \beta^t a_t$.

The sequences $(a_0, .., a_N, c_{N+1}, ..., c_t, ..)$ and $(a'_0, ..., a'_N, c'_{N+1}, ..., c'_{N+\tau}, ..)$ belong to l^∞_+. We have

$$\sum_{t=0}^{N} \beta^t a_t + \sum_{t=N+1}^{+\infty} \beta^t c_t < \sum_{t=0}^{N} \beta^t a'_t + \sum_{t=N+1}^{+\infty} \beta^t c'_t.$$

That shows the present is not dictatorial.

However, when the discounted utilitarian criterion is used in an optimal growth model such as ours, the present becomes dictatorial. Indeed, given k_0 large enough, the feasible capital stocks will be bounded by $\|k_0\|$. This implies the feasible consumptions be bounded by some $A(k_0)$ (see Le Van and Dana [2003], chap. 4). The proof given above applies.

3. We now give two examples of non-dictatorial criteria.

3.1. *Rawls' criterion*
Let

$$W(\mathbf{a}) = \inf_t a_t,\ a_t \geq 0,\ \forall t.$$

We claim that the present is NOT dictatorial for this criterion.
Let $W(\mathbf{a}) > W(\mathbf{b})$. Take $c_t = 0,\ \forall t$. For any n, define $d(n)_t = a_t,\ \ \text{for } t \leq n$, and $d(n)_t = c_t,\ \forall t > n$. Then $0 = W(\mathbf{d(n)}) \leq W(\mathbf{b})$.
We now show that the future is NOT dictatorial.
Again, let $W(\mathbf{a}) > W(\mathbf{b})$. Take $c_t = 0,\ \forall t$. For any n, define $d(n)_t = c_t,\ \ \text{for } t \leq n$, and $d(n)_t = a_t,\ \forall t > n$. Then $0 = W(\mathbf{d(n)}) \leq W(\mathbf{b})$.
3.2. *Chichilnisky's Criterion*

$$W(\mathbf{a}) = \sum_{t=0}^{+\infty} \beta^t a_t + \phi(\mathbf{a}),$$

where ϕ is purely additive, increasing, i.e. ϕ is an increasing continuous linear form on l^∞, $\phi(\mathbf{a}) = 0$ if $\{t : a_t \neq 0\}$ is finite, and hence $\phi(\mathbf{a}) = 0$ if $a_t \to 0$ when $t \to +\infty$. If ϕ is well chosen then this criterion is not dictatorial, neither for the present nor for the future.
However, we give an example where the Chichilnisky's criterion is dictatorial for the present when ϕ is not well chosen.
Assume $\phi(1) < \frac{1}{1-\beta}$. Let $\mathbf{a} = \mathbf{1}$ and $\mathbf{b}$ satisfy $\sum_{t=0}^{+\infty} \beta^t b_t + \phi(\mathbf{1}) < \frac{1}{1-\beta}$. Then

$$\sum_{t=0}^{+\infty} \beta^t b_t + \phi(\mathbf{b}) \leq \sum_{t=0}^{+\infty} \beta^t b_t + \phi(\mathbf{1}) < \sum_{t=0}^{+\infty} \beta^t a_t + \phi(\mathbf{a}).$$

We have, for any $(c_t)_t : c_t \geq 0,\ \forall t,\ \sup_t c_t \leq 1$, any $(d_t)_t : d_t \geq 0,\ \forall t,\ \sup_t d_t \leq 1$, for any n large enough

$$\begin{aligned}
& \sum_{t=0}^{n} \beta^t a_t + \sum_{t=n+1}^{+\infty} \beta^t c_t + \phi(a_0, ...a_n, c_{n+1}, ...) \\
\geq\ & \sum_{t=0}^{n} \beta^t a_t = \frac{1}{1-\beta} - \frac{\beta^{n+1}}{1-\beta} \\
>\ & \sum_{t=0}^{n} \beta^t b_t + \sum_{t=n+1}^{+\infty} \beta^t d_t + \phi(\mathbf{1}) \\
\geq\ & \sum_{t=0}^{n} \beta^t b_t + \sum_{t=n+1}^{+\infty} \beta^t d_t + \phi(b_0, ..., b_n, d_{n+1}, ...)
\end{aligned}$$

4. We consider again the economy set in Section 2. Let us consider the problem

$$\max \lim_{T \to +\infty} \sum_{t=0}^{T} [u(x_t, x_{t+1}) - u(\overline{x}, \overline{x})]$$

under the constraints

$$\forall t,\ x_{t+1} \in \Gamma(x_t),\ x_0 \text{ is given.}$$

We will show that this problem can be approximated as follows. Assume $G(x_0) \neq \emptyset$. Let $\mathbf{x}^*$ be optimal from x_0. We know that $x_t^* \to \overline{x}$. since $(\overline{x}, \overline{x}) \in$ intgraphΓ, there exists T_0 such that for any $t \geq T_0$, $\overline{x} \in \Gamma(x_t^*)$. Let consider the problem, for $T \geq T_0$,

$$\max \sum_{t=0}^{T} [u(x_t, x_{t+1}) - u(\overline{x}, \overline{x})]$$

under the constraints

$$\forall t,\ x_{t+1} \in \Gamma(x_t),\ x_0 \text{ is given and } x_{T+1} = \overline{x}.$$

Let $(\widehat{x}_t(T))_{t=0}^{T}$ be the optimal solution. Let

$$\widehat{\mathbf{x}}(\mathbf{T}) = (x_0, \widehat{x}_1(T), ..., \widehat{x}_T(T), \overline{x}, \overline{x}, ...)$$

We have

$$\sum_{t=0}^{T} [u(\widehat{x}_t(T), \widehat{x}_{t+1}(T)) - u(\overline{x}, \overline{x})] \geq \sum_{t=0}^{T-1} \left[u(x_t^*, x_{t+1}^*) - u(\overline{x}, \overline{x})\right] + [u(x_T^*, \overline{x}) - u(\overline{x}, \overline{x})].$$

and

$$\begin{aligned}
\sum_{t=0}^{+\infty} [u(\widehat{x}_t(T), \widehat{x}_{t+1}(T)) - \overline{u}] &= \sum_{t=0}^{T} [u(\widehat{x}_t(T), \widehat{x}_{t+1}(T)) - \overline{u}] \\
&= \sum_{t=0}^{T-1} [u(\widehat{x}_t(T), \widehat{x}_{t+1}(T)) - \overline{u}] + [u(\widehat{x}_T(T), \overline{x}) - \overline{u}] \\
&\geq \sum_{t=0}^{T-1} \left[u(x_t^*, x_{t+1}^*) - \overline{u}\right] + [u(x_T^*, \overline{x}) - \overline{u}]
\end{aligned}$$

The sequence $\widehat{\mathbf{x}}(\mathbf{T})$ converges (for the product topology) to $\widehat{\mathbf{x}}$ when $T \to +\infty$. ¿From proposition 3, and since $x_T^* \to \overline{x}$, one gets

$$\begin{aligned}
\sum_{t=0}^{+\infty} [u(\widehat{x}_t, \widehat{x}_{t+1}) - \overline{u}] &\geq \limsup_T \sum_{t=0}^{+\infty} [u(\widehat{x}_t(T), \widehat{x}_{t+1}(T)) - \overline{u}] \\
&\geq \sum_{t=0}^{+\infty} \left[u(x_t^*, x_{t+1}^*) - \overline{u}\right]
\end{aligned}$$

Since $\widehat{x}_{t+1} \in \Gamma(\widehat{x}_t),\ \forall t$, we have $\widehat{\mathbf{x}} = \mathbf{x}^*$ since u is strictly concave. In other words, the sequences $\widehat{\mathbf{x}}(\mathbf{T})$ approximate $\mathbf{x}^*$.

5. We show that our methodology can be used for an economy with exogenous growth. Consider for instance an economy where for each period t, one has

$$C_t + K_{t+1} - (1-\delta)K_t \leq (1+g)^{(1-\alpha)t} K_t^{\alpha},\ 0 < \alpha < 1.$$

Let $c_t = \frac{C_t}{(1+g)^t},\ k_t = \frac{K_t}{(1+g)^t}$. One obtains

$$c_t + k_{t+1}(1+g) \leq k_t^{\alpha} + (1-\delta)k_t.$$

Denote by f the function $f(k) = k^{\alpha} + (1-\delta)k$. Let v be the instantaneous utility function and $u(k,y) = v(f(k) - (1+g)y)$. Let $\overline{u} = u(\overline{k}, \overline{k})$ where $\overline{k}$ maximizes $\{f(k) - (1+g)k : 0 \leq k(1+g) \leq f(k)\}$. Observe that $F'(\overline{k}) = g + \delta$, where $F(k) = k^{\alpha}$. Our model will be

$$\max \sum_{t=0}^{+\infty} [u(k_t, k_{t+1}) - \overline{u}]$$

under the constraints

$$\forall t,\ 0 \leq (1+g)k_{t+1} \leq f(k_t),\ k_0 \text{ is given.}$$

6. Our criterion does not require the state space X to be compact, while the Chichilnisky and the Basu-Mitra criteria require a model with a compact state space. One can criticize that our criterion excludes the consumption paths (c_t) which converge to zero. But observe, for such a situation, the value of the stream of utilities generated by the discounted utilitarian criterion is the same as the one generated by Chichilnisky's criterion since $\phi(\mathbf{u}) = 0$, where $u_t = u(c_t),\ \forall t$. For a sustainable growth, it is reasonable to have optimal paths which converge to a steady state which can be interpreted as the Golden Rule.

7. Let $\mathbf{x}^*$ be an optimal path from x_0 in our model. Assume $G(x_0) \neq \emptyset$. Let $\zeta_t^* = u(x_t^*, x_{t+1}^*), \forall t$. For two feasible sequences $\mathbf{x}, \mathbf{x}'$, we write $\zeta_t = u(x_t, x_{t+1}),\ \zeta_t' = u(x_t', x_{t+1}')$. Following Basu and Mitra [2007], we write $\boldsymbol{\zeta} \succeq_U \boldsymbol{\zeta}'$ if there exists N such that

$$(\sum_{t=0}^{N}(\zeta_t - \overline{u}), \zeta_{N+1} - \overline{u}, ..., \zeta_{N+\tau} - \overline{u}, ...) \geq (\sum_{t=0}^{N}(\zeta_t' - \overline{u}), \zeta_{N+1}' - \overline{u}, ..., \zeta_{N+\tau}' - \overline{u}, ...)$$

In particular, if there exists N such that

$$(\sum_{t=0}^{N}(\zeta_t - \overline{u}), \zeta_{N+1} - \overline{u}, ..., \zeta_{N+\tau} - \overline{u}, ...) > (\sum_{t=0}^{N}(\zeta_t' - \overline{u}), \zeta_{N+1}' - \overline{u}, ..., \zeta_{N+\tau}' - \overline{u}, ...)$$

then $\boldsymbol{\zeta} \succ_U \boldsymbol{\zeta}'$. The Social Welfare Relation $\succeq_U$ is called utilitarian Social Welfare Function. Using the version given by Atsumi [1965], von Weizsacker [1965], and Brock [1970b], they define an overtaking Social Welfare Relation as follows:

$$\boldsymbol{\zeta} \succeq_O \boldsymbol{\zeta}' \text{ iff}$$

- either $\exists \bar{N}$, such that $\sum_{t=0}^{N}(\zeta_t - \overline{u}) > \sum_{t=0}^{N}(\zeta'_t - \overline{u})$ for all $N \geq \bar{N}$,
- or $\exists \bar{N}$ such that $\sum_{t=0}^{N}(\zeta_t - \overline{u}) = \sum_{t=0}^{N}(\zeta'_t - \overline{u})$ for all $N \geq \bar{N}$.

Hence, $\boldsymbol{\zeta} \succ_O \boldsymbol{\zeta}'$ iff there exists $\bar{N}$ such that $\sum_{t=0}^{N}(\zeta_t - \overline{u}) > \sum_{t=0}^{N}(\zeta'_t - \overline{u})$ for all $N \geq \bar{N}$.
We say that a feasible path $\mathbf{x}$ is maximal if there exists no feasible $\mathbf{x}'$ with $\mathbf{x}' \succ_U \mathbf{x}$. For the one-dimension Ramsey model, Basu and Mitra [2007] show that if $\mathbf{x}^*$ is maximal then for any feasible path $\mathbf{x}'$, we have $\boldsymbol{\zeta}^* \succ_O \boldsymbol{\zeta}'$. In our paper, actually we extend their result to a multi-dimensional Ramsey model. Indeed, from theorem 4, there exists an optimal path $\mathbf{x}^*$. We claim that it is maximal. If not, there will be $\mathbf{x}$ and N such that

$$(\sum_{t=0}^{N}(\zeta_t - \overline{u}), \zeta_{N+1} - \overline{u}, ..., \zeta_{N+\tau} - \overline{u}, ...) > (\sum_{t=0}^{N}(\zeta_t^* - \overline{u}), \zeta_{N+1}^* - \overline{u}, ..., \zeta_{N+\tau}^* - \overline{u}, ...)$$

But in this case,

$$\sum_{t=0}^{+\infty}(\zeta_t - \overline{u}) > \sum_{t=0}^{+\infty}(\zeta_t^* - \overline{u})$$

contradicting the optimality of $\mathbf{x}^*$.
Now assume that graphΓ is convex. In this case, for any feasible $\mathbf{x}$ different from $\mathbf{x}^*$

$$\sum_{t=0}^{+\infty}[u(x_t^*, x_{t+1}^*) - \overline{u}] > \sum_{t=0}^{+\infty}[u(x_t, x_{t+1}) - \overline{u}]$$

We claim that $\boldsymbol{\zeta}^* \succ_O \boldsymbol{\zeta}$. Indeed, if there exists no $\bar{N}$ such that

$$\sum_{t=0}^{N}[u(x_t^*, x_{t+1}^*) - \overline{u}] > \sum_{t=0}^{N}[u(x_t, x_{t+1}) - \overline{u}]$$

for any $N \geq \bar{N}$, there will be an increasing sequence $(N_k)_k$ such that

$$\sum_{t=0}^{N_k}[u(x_t^*, x_{t+1}^*) - \overline{u}] \leq \sum_{t=0}^{N_k}[u(x_t, x_{t+1}) - \overline{u}]$$

for every k. This implies

$$\sum_{t=0}^{+\infty}[u(x_t^*, x_{t+1}^*) - \overline{u}] \leq \sum_{t=0}^{+\infty}[u(x_t, x_{t+1}) - \overline{u}]$$

and $\mathbf{x}$ is another optimal path contradicting theorem 4.

8. Fleurbaey and Michel [2003] prove that there exists no social welfare relation which is together weak Pareto, indifferent to finite permutations and strongly continuous (i.e. continuous for the sup-norm topology). In our paper, the social welfare function is Pareto, indifferent to finite permutations but only upper semi-continuous for the product topology. Observe there is a parallel between their extended Ramsey criterion which is $\lim_{T\to+\infty}\sum_{t=0}^{T}[u(c_t)-\widehat{u}]$ where c_t denotes the consumption at date t and $\widehat{u}$ is the "bliss point", $\widehat{u}=\lim_{x\to+\infty}u(x)$, and our criterion $\lim_{T\to+\infty}\sum_{t=0}^{T}[u(x_t,x_{t+1})-\overline{u}]$, since $\overline{u}=\lim_{t\to+\infty}u(x_t,x_{t+1})$ (any good programme converges).

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