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Abstract

The international asset pricing models are mostly developed in the case of parity failure (investors of different countries do not agree on the expected returns on securities). In this case, an equilibrium in the international asset markets may exist, but not in the international good markets. In our paper, we prove the existence of an equilibrium in both the asset and the good markets. We focus also on the links between parities, no-arbitrage conditions and the general equilibrium. We show that no-arbitrage conditions for international asset and good markets are necessary and sufficient to an equilibrium in both the markets.

Keywords: international asset pricing, returns on securities, exchange rates, no-arbitrage conditions, PPP, UIRP, general equilibrium.

JEL Classification: D53, F31, G11, G15.

1   Introduction

Most of the international asset pricing models rest on partial equilibrium arguments. The exchange rates are exogenous and the purchasing power parity or the interest rate parity are violated. Deviations from the uncovered interest rate parity due to different risk perception, limited market participation or bounded rationality imply that national groups of investors evaluate differently the physical returns on the same security. When this happens, a disequilibrium of the trade balance may coexist with an equilibrium in the asset markets.

We revisit a two-period financial model by Hart [11] to take in account the international trade. In the first period, agents buy or sell financial assets to diversify their portfolios and maximize an expected utility function. In the second period, they buy or sell goods with their initial endowments and the gains from financial investments. Contrarily to Solnik [17], we allow agents to exchange goods across the borders. Security returns and goods are valued in domestic currencies.

In this framework, we prove the existence of an equilibrium in the asset markets jointly with an equilibrium of the trade balance in the international good markets. Characterizations of equilibrium existence are also provided in terms of no arbitrage.

Two notions of no arbitrage are considered. The first one is usual in the financial literature and refers to the existence of appropriate asset prices that prevent agents from making gains that outpace market gains without taking on more risk. The second one rests on the existence of exchange rates that prevent agents from buying some consumption good from one country and reselling to another to make gains.

More precisely, on the side of goods, we show the equivalence between the uncovered interest rate parity[1] and a no-arbitrage condition in the international good market. On the side of assets, we prove the equivalence of different no-arbitrage conditions in the financial markets.

In order to prove the existence and optimality properties of equilibrium, we distinguish between the notions of equilibrium and equilibrium with transfer. Parities and no-arbitrage conditions are found to be fundamental equilibrium properties. The difficult question of sufficiency of these conditions is also raised.

No-arbitrage conditions ensures an equilibrium in the international financial markets. Our paper differs from Solnik [17] where the purchasing power parity is not respected and the issue of existence of a general equilibrium is not addressed.

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[1] The Uncovered Interest Rate Parity (UIRP) is a well-known no-arbitrage condition. The Interest Rate Parity (IRP) between two countries is an equilibrium no-arbitrage condition such that investors become indifferent to interest rates available in these countries. IRP rests on perfect mobility and substitutability of financial assets. UIRP is an IRP holding when investors are indifferent among the available interest rates in two countries because the exchange rate is expected to adjust such that the return on the domestic asset is equal to that on foreign assets valued in domestic currency. UIRP eliminates the potential for uncovered interest arbitrage profits (see Hallwood and MacDonald [10] among others). In dynamic models, UIRP means that the difference in the returns at home and abroad is equal to the relative (expected) depreciation in the exchange rate.

It differs also from Dumas [6] who pays attention only to Pareto allocations in a two-country economy where agents are prevented from consuming foreign goods. In the spirit of Ross and Walsh [16], we find that the purchasing power parity holds in the good markets when suitable conditions on security returns are met.

Our paper also extends Dana, Le Van and Magnien [5] where, under strictly concave utility functions, no-arbitrage conditions in the financial markets are found to be necessary and sufficient to the existence of an equilibrium in these markets. Indeed, we introduce an additional condition of no arbitrage in the international good markets and we prove that, in the risk-neutral case, no-arbitrage conditions for asset and good markets characterize the existence of a general equilibrium.

Following Dumas [7], in the case of risk aversion, we give an example of nonexistence of equilibrium in the international good markets when the no-arbitrage condition in the international good markets fails. Under risk neutrality, we show that any net trade is an equilibrium with transfer if and only if this no-arbitrage condition in the international good markets and no-arbitrage conditions in the asset markets hold. Under UIRP failure, a second example shows that the no-trade equilibrium is the unique equilibrium in the case of risk neutrality.

Summing up, we highlight a possible source of imbalances in the international trade with financial assets. Imbalances **may** arise when the system of returns for trading countries does not meet the no-arbitrage conditions together, that is the condition for asset markets and that for international good markets.

The paper is organized as follows. We introduce the main assumptions of the model in Section 2 and the no-arbitrage conditions, either international or financial, in Section 3. In Section 4, the question of equilibrium existence is separately addressed for a consumption good and a wealth model (where wealth is valued instead of consumption by agents). An example of failure of the uncovered interest rate parity is considered and the possibility of an equilibrium in the asset markets with a disequilibrium in the good markets is highlighted. The paper ends with the issue of equilibrium existence in the risk-neutral case. Section 6 concludes.

2   The model

Focus on a pure exchange economy where assets and goods are traded in international markets. Before addressing the equilibrium issue, let us introduce notations and assumptions, and describe saving diversification and consumption.

We consider a two-period exchange economy with many countries. Financial assets are traded in the first period and a good is consumed in the second. The representative agent of country $i \in \{0, \ldots, I\}$ purchases K assets in period 0 to smooth consumption in period 1 across S states of nature. Nature provides an endowment in period 1.

2.1 Notations

Because of the complexity of the model, many variables are involved and a well-ordered notation is needed. Let us provide the entire set of notations before entering the mechanics of the model.

We introduce first a compact notation for asset prices and quantities on the financial side of the economy.

$q \equiv (q_1, \ldots, q_K)$ is a row of asset prices where q_k denotes the price of asset k.

$x \equiv \left(x_k^i\right)$ is the $K \times (1 + I)$ matrix of portfolios where x_k^i denotes the amount of asset k in the portfolio of agent i. Column $x^i \equiv \left(x_1^i, \ldots, x_K^i\right)^T$ is the portfolio of agent i.

$R^i \equiv \left(R_{sk}^i\right)$ is the $S \times K$ matrix of returns where $R_{sk}^i \geq 0$ denotes the return[2] on asset k in the state of nature s. R_s^i is the sth row of the matrix. Returns R_{sk}^i are valued in the currency of country i.

We introduce now a compact notation for beliefs, prices and quantities on the real side of the economy.

$\pi \equiv \left(\pi_s^i\right)$ is the $(1 + I) \times S$ matrix of beliefs where π_s^i denotes the belief of agent i about the occurrence of state s. The individual row of beliefs $\pi^i \equiv \left(\pi_1^i, \ldots, \pi_S^i\right)$ lies in the S-unit simplex.

$p \equiv \left(p_s^i\right)$ is the $(1 + I) \times S$ matrix of good prices where p_s^i denotes the price of the good in country i in the state of nature s. $p^i \equiv \left(p_1^i, \ldots, p_S^i\right)$ is the ith row of the matrix.

$\tau \equiv \left(\tau_s^i\right)$ is the $(1 + I) \times S$ matrix of exchange rates where τ_s^i denotes the exchange rate between currencies of country 0 and country i in the state of nature s. $\tau^i \equiv \left(\tau_1^i, \ldots, \tau_S^i\right)$ is the ith row of the matrix. The first row is a vector of units: $\tau_s^0 = 1$ for any s.

$c \equiv \left(c_s^i\right)$ is the $S \times (1 + I)$ matrix of consumptions where c_s^i denotes the amount of good consumed by agent i in the state of nature s. $c^i \equiv \left(c_1^i, \ldots, c_S^i\right)^T$ is the consumption column of agent i. The amount c_s^i is valued in the currency of country i.

$e \equiv \left(e_s^i\right)$ is the $S \times (1 + I)$ matrix of endowments where e_s^i denotes the endowment nature provides to agent i in the state s. $e^i \equiv \left(e_1^i, \ldots, e_S^i\right)^T$ is the endowment column of agent i. The endowment e_s^i is valued in the currency of country i.

Notice that prices and beliefs q, R_s^i, τ^i, p^i, π^i are rows, while quantities x^i, c^i, e^i are columns.

The individual consumption value is given by $p^i c^i$. Since c_s^i is valued in the currency of country i, we interpret p_s^i as the inverse of an ordinary price and $p_s^i c_s^i$ as the physical value of c_s^i. The physical values can be aggregated over the states in a physical value of consumption $p^i c^i$.

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[2]The return is the value of one unit of security in the second period including the dividend. Agents form beliefs about the future and associate to each return the probability of its state of nature.

In the article, $\sum_i$, $\sum_s$, $\sum_k$ will denote unambiguously the explicit sums $\sum_{i=0}^{I}$, $\sum_{s=1}^{S}$, $\sum_{k=1}^{K}$.

2.2 Assumptions

In order to prove and characterize the existence of a general equilibrium with financial assets, we introduce some mild assumptions. The first triplet of hypotheses specifies the financial fundamentals (returns); the second triplet specifies the real fundamentals (endowments and preferences).

Assumption 1 *For any country i and any state s, $\sum_k R_{sk}^i > 0$.*

Assumption 2 *For any country i and any asset k, $\sum_s R_{sk}^i > 0$.*

When Assumption 1 fails, there is a country i and a state s where any asset k yields $R_{sk}^i = 0$. In this case, the representative agent of country i will consume her endowment in the state s.

When Assumption 2 fails, there is an asset k yielding $R_{sk}^i = 0$ in any state of nature s in country i: the representative agent i will refuse to buy this asset. The following assumption is stronger and implies Assumption 2.

Assumption 3 *For any country i and any portfolio $x^i \neq 0$, the portfolio return is nonzero: $R^i x^i \neq 0$.*

Assumption 3 means that there are no nonzero portfolios with a null return in any state of nature. In other terms, whatever the country i we consider, rank$R^i = K$ and the K assets are not redundant.[3]

Assumption 4 *Endowments are positive: $e_s^i > 0$ for any agent i and any state s.*

Assumption 5 *Beliefs are positive: $\pi_s^i > 0$ for any agent i and any state s.*

This assumption means that any representative agent thinks each state as possible.

Eventually, preferences are required to satisfy regular assumptions. We distinguish the case where consumption is valued by agents from that where wealth is valued.

Assumption 6 *For any agent i, the utility function u^i is differentiable, strictly increasing and concave in $\mathbb{R}_{++}$ for the consumption model, and in $\mathbb{R}$ for the wealth model.*

2.3 Portfolios

Agents' behavior comes down to a saving diversification to finance future consumption. In the state s, agents exchange their endowments according to their portfolio:

$$c_s^i = e_s^i + R_s^i x^i \tag{1}$$

---

[3]Markets completeness means that the columns of R^i span the whole space $\mathbb{R}^S$ (rank$R^i = S$) and implies that a full insurance is possible. Redundancy of assets means that $\dim \ker R^i > 0$, that is $K > $ rankR^i. When markets are complete and assets are not redundant, we have $K = S = $ rankR^i. In this case, the return matrix is square and invertible.

Preferences of agent i are rationalized by a Von Neumann-Morgenstern utility function weighted by subjective probabilities: $\sum_s \pi_s^i u^i \left(c_s^i\right)$, where u^i is the utility function and c_s^i can be interpreted as consumption or wealth.

As a consumption amount, c_s^i is required to be nonnegative and the utility function is defined on the nonnegative orthant. As wealth, c_s^i is permitted to become negative in some state of nature and the utility function is defined on the whole space.

These two cases correspond also to different portfolio sets and will be referred as the consumption and the wealth model respectively.

(1) Consumption model. We requires $c_s^i \geq 0$ for any agent i in any state of nature s. The portfolio set writes

$$X^i \equiv \left\{x^i \in \mathbb{R}^K : for any s,\ e_s^i + R_s^i x^i \geq 0\right\} \tag{2}$$

(2) Wealth model. Wealth is allowed to be negative: $c_s^i \in \mathbb{R}$, and the portfolio set X^i coincides with the whole space $\mathbb{R}^K$.

In the first period, agent i diversifies her portfolio picking a vector in the portfolio set and taking in account the financial budget constraint:

$$\max_{x^i \in X^i} \sum_s \pi_s^i u^i \left(e_s^i + R_s^i x^i\right) \tag{3}$$
$$qx^i \leq 0$$

where the portfolio set X^i depends on the model we consider. The RHS of the budget constraint is zero because agents enter the financial market with no asset endowments. Equivalently, we consider only the agents' net purchases.

Before proving the existence of an equilibrium in both the international good and asset markets, let us provide the no-arbitrage conditions associated to this equilibrium.

3 Arbitrages

In economics, arbitrage is the practice of taking advantage of a price difference between two markets. Thereby, there is room for arbitrage when the law of one price is violated. In the international trade literature, arbitrage is possible when the same good has different prices in different countries expressed in the same currency. In finance, arbitrage is possible when the same asset does not trade at the same price in two markets. By definition, no-arbitrage conditions are sufficient to rule out any profitable arbitrage in the asset markets, but they are also necessary to the existence of a general equilibrium.

Let us focus on two classes of conditions to exclude arbitrage opportunities: parities, when goods are traded across the borders, and no-arbitrage conditions, when assets are exchanged.

3.1 International arbitrage

In this section, we consider the international trade and parities as specific no-arbitrage conditions for good markets.

Two notions of parity are usually applied in the theory of international trade: the Purchasing Power Parity (PPP) and the Uncovered Interest Rate Parity (UIRP).

The PPP is equivalent to the law of one price. In the absence of transactions costs and trade barriers, the same good has the same price in different countries when prices are expressed in the same currency.[4]

The Uncovered Interest Rate Parity (UIRP) holds when investors are indifferent among the available interest rates in two countries because the exchange rate is expected to adjust such that the return on domestic assets is equal to that on foreign assets valued in domestic currency.

Condition 1 *(PPP) Given an exchange rate system τ^* with $\tau_s^{0*} = 1$, the price system p satisfies the PPP if $p_s^i = \tau_s^{i*} p_s^0$ for any country i and any state s.*

Condition 2 *(UIRP) There exists a matrix τ^* such that, for any agent i and any state s, we have $R_s^0 = \tau_s^{i*} R_s^i$ with $\tau_s^{0*} = 1$.*

Our condition (UIRP) extends the usual PPP or the law of one price. To see that, consider an economy with two countries, 0 and 1. Suppose that the first asset is riskless for both countries, i.e. $R_{s1}^1 = R_{s1}^0 = 1$ for any state s. If (UIRP) holds, then $\tau_s^{1*} = 1$ for every s. We then have $R_{sk}^1 = R_{sk}^0$ for any s, any k. In this case, at equilibrium $p_s^1 = p_s^0$ for any s.

Parities rule out profitable exchanges and, in this sense, work as no-arbitrage conditions for physical markets. To clarify this point, we introduce an explicit no-arbitrage condition in the international good markets or, in short, NAG (no-arbitrage condition for goods) and we prove its equivalence with UIRP. NAG rests on the notion of net trade.

Definition 3 *(individually rational allocation) A matrix of portfolios x is individually rational if $\sum_s \pi_s^i u^i \left(e_s^i + R_s^i x^i \right) \geq \sum_s \pi_s^i u^i \left(e_s^i \right)$ for any i.*

Definition 4 *(net trade) A matrix of portfolios x is a net trade if $\sum_i x^i = 0$. A net trade x is individually rational if x is an individually rational allocation.*

We say that there exists an arbitrage in the international good markets if, for any price system p with $p_s^i > 0$ for any i and s, there exists a net trade x such that $\sum_i p_s^i R_s^i x^i \neq 0$ (that is $\sum_i p_s^i c_s^i \neq \sum_i p_s^i e_s^i$) for some s where $R_s^i x^i$ is the return on the portfolio of agent i in the state of nature s.

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[4]The relative version of the PPP states that the difference in the (expected) inflation rates abroad and at home is equal to the relative (expected) appreciation in the exchange rate. In our paper, good prices refer only to the second period and the relative PPP turns out to be of no use.

Condition 5 *(NAG) There exists a price system p with $p_s^i > 0$ for any i and s such that, for any net trade x, we have $\sum_i p_s^i R_s^i x^i = 0$ for any s.*

Proposition 6 *NAG is equivalent to UIRP.*

Proof. Let NAG be satisfied. Fix a pair (i, k) and define $x_k^0 = -1$, $x_k^i = 1$ and $x_h^j = 0$ if $(j, h) \neq (i, k)$. By definition, x is a net trade. By NAG, there exists a price system p such that $\sum_j p_s^j R_s^j x^j = 0$ for any s. This implies $p_s^i R_{sk}^i - p_s^0 R_{sk}^0 = 0$ for any s. Define $\tau_s^{i*} \equiv p_s^i/p_s^0$. We get $R_{sk}^0 = R_{sk}^i \tau_s^{i*}$. This holds for any choice of pair (i, k) and any state of nature s. Thus, UIRP is satisfied.

Conversely, assume that the UIRP holds. Thus, there exists a matrix τ^* such that $R_s^0 = \tau_s^{i*} R_s^i$ for any i and any s. Set $p_s^i \equiv \tau_s^{i*}$ for any i and any s and consider a net trade x. We obtain $\tau_s^{i*} R_s^i x^i = R_s^0 x^i$ and, summing over i, $\sum_i p_s^i R_s^i x^i = \sum_i \tau_s^{i*} R_s^i x^i = R_s^0 \sum_i x^i = 0$. Then, the price system $p \equiv \tau^*$ satisfies the NAG. ∎

3.2 Financial arbitrage

Another class of no-arbitrage conditions are introduced to characterize the financial side of equilibrium. No-arbitrage conditions NA0 and NA1 require the existence of asset price systems that, respectively, make useful (utility-improving) portfolios and portfolios with nonnegative returns beyond agents' reach. Under mild assumptions, conditions NA0 and NA1 are equivalent.

Definition 7 *(useful vector) y is a useful vector for a function $f : \mathbb{R}^n \to \mathbb{R}$ evaluated in x if $f(x + \mu y) \geq f(x)$ for any $\mu \geq 0$.*

Definition 8 *(useful portfolio) w^i is a useful portfolio[5] for agent i if, for any $\mu \geq 0$ and any $x^i \in X^i$, one has:*
(1) $x^i + \mu w^i \in X^i$,
(2) $\sum_s \pi_s^i u^i \left(e_s^i + R_s^i \left(x^i + \mu w^i \right) \right) \geq \sum_s \pi_s^i u^i \left(e_s^i + R_s^i x^i \right)$.
Let $V^i \equiv \left\{ v^i \in \mathbb{R}^K : R_s^i v^i \geq 0 \text{ for any } s \right\}$ be the set of portfolios with nonnegative returns in any state of nature and W^i denote the set of useful portfolios for agent i.

Notice that the notion of useful portfolio is different from that of useful vector, while the sets V^i and W^i are closely related.

Proposition 9 *Let Assumption 6 hold. In the consumption model, for any agent i, $V^i = W^i$, while, in the wealth model, $V^i \subseteq W^i$.*

Proof. Focus on the consumption model.

First, we want to prove that $V^i \subseteq W^i$. If $v^i \in V^i$, $R_s^i v^i \geq 0$ for any s. According to (2), for any $x^i \in X^i$ and any $\mu \geq 0$, one has $e_s^i + R_s^i \left(x^i + \mu v^i \right) = e_s^i + R_s^i x^i + \mu R_s^i v^i \geq \mu R_s^i v^i \geq 0$, that is Definition 8, point (1). From

$$e_s^i + R_s^i \left(x^i + \mu v^i \right) = e_s^i + R_s^i x^i + \mu R_s^i v^i \geq e_s^i + R_s^i x^i \tag{4}$$

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[5] For a definition of useful and useless purchases, see Werner [18] among others.

and the increasingness of u^i (Assumption 6), we obtain also point (2).

Conversely, we want to show that $W^i \subseteq V^i$. Let $w^i \in W^i$. Then, point (1) in Definition 8 requires $e_s^i + R_s^i \left(x^i + \mu w^i \right) \geq 0$ for any s. Dividing both the sides of this inequality by $\mu > 0$ and letting μ go to infinity, we get $R_s^i w^i \geq 0$ for any s, that is $w^i \in V^i$.

In the wealth model, it is immediate to prove that $V^i \subseteq W^i$: indeed, point (1) of Definition 8 holds because $x^i + \mu w^i \in \mathbb{R}^K$, while point (2) still rests on (4) and increasingness of u^i. The converse does not hold in general. ∎

Proposition 10 *In the wealth model, under Assumption 6, a vector w^i is useful for agent i if and only if $\sum_s \pi_s^i u^{i\prime} \left(e_s^i + R_s^i x^i \right) R_s^i w^i \geq 0$ for any $x^i \in \mathbb{R}^K$.*

Proof. As in Dana and Le Van [3], [4], the proof rests on the concavity and differentiability of u^i. Notice that $\sum_s \pi_s^i u^{i\prime} \left(e_s^i + R_s^i x^i \right) R_s^i w^i$ is the derivative of $\sum_s \pi_s^i u^i \left(e_s^i + R_s^i \left(x^i + \mu w^i \right) \right)$ with respect to μ in Definition 8, point (2). ∎

Dana and Le Van [4] provide another characterization of the set of useful vectors W^i in the case of wealth model. Their argument can be adapted as follows.

Proposition 11 *Consider the wealth model. Let $\rho_s^i \equiv R_s^i w^i$ be the return on the portfolio $w^i \in X^i$ in the state of nature s. Under Assumptions 3 and 6, a vector w^i is useful for i if and only if*

$$a^i \sum_{s \in S_+^i} \pi_s^i \rho_s^i + b^i \sum_{s \in S_-^i} \pi_s^i \rho_s^i \geq 0 \qquad (5)$$

where $a^i \equiv u^{i\prime} \left(+\infty \right)$ and $b^i \equiv u^{i\prime} \left(-\infty \right)$ denote the asymptotic derivatives, while $S_-^i \equiv \left\{ s : \rho_s^i < 0 \right\}$ and $S_+^i \equiv \left\{ s : \rho_s^i \geq 0 \right\}$ the sets of states with negative and nonnegative returns respectively.

Proof. Let w^i be a useful portfolio for agent i. By Definition 8,

$$\sum_s \pi_s^i u^i \left(e_s^i + R_s^i \left(x^i + \mu w^i \right) \right) \geq \sum_s \pi_s^i u^i \left(e_s^i + R_s^i x^i \right)$$

for any $\mu \geq 0$ and any $x^i \in \mathbb{R}^K$. $c^i = e^i + R^i x^i$ implies $\sum_s \pi_s^i u^i \left(c_s^i + \mu \rho_s^i \right) \geq \sum_s \pi_s^i u^i \left(c_s^i \right)$ for any $\mu \geq 0$. According to Definition 5, $\rho^i \equiv \left(\rho_1^i, \dots, \rho_S^i \right)^T$ is a useful vector for the utility function $U^i \left(c^i \right) \equiv \sum_s \pi_s^i u^i \left(c_s^i \right)$ evaluated in c^i. Because of the concavity of u^i (Assumption 6), for $\mu = 1$, we obtain $\sum_s \pi_s^i u^{i\prime} \left(c_s^i \right) \rho_s^i \geq \sum_s \pi_s^i u^i \left(c_s^i + \rho_s^i \right) - \sum_s \pi_s^i u^i \left(c_s^i \right) \geq 0$. Letting c_s^i go to $+\infty$ ($-\infty$) for any $s \in S_+^i$ ($s \in S_-^i$), we obtain (5). We notice that any c_s^i can go to $+\infty$ or $-\infty$ because rank$R^i = K$ (Assumption 3).

Conversely, let (5) hold. Then, by Assumption 6,

$$
\begin{aligned}
0 \quad &\leq \quad \sum_{s \in S_+^i} a^i \pi_s^i \rho_s + \sum_{s \in S_-^i} b^i \pi_s^i \rho_s \leq \sum_{s \in S_+^i} u^{i\prime} \left(c_s^i \right) \pi_s^i \rho_s + \sum_{s \in S_-^i} u^{i\prime} \left(c_s^i \right) \pi_s^i \rho_s \\
&= \quad \sum_s \pi_s^i u^{i\prime} \left(c_s^i \right) \rho_s = \sum_s \pi_s^i u^{i\prime} \left(e_s^i + R_s^i x^i \right) R_s^i w^i
\end{aligned}
$$

Eventually, Proposition 10 applies. ∎

Thus, the set of useful portfolios W^i is larger for the wealth model because it includes the set of useful portfolios V^i of the consumption model. However, under an additional assumption, the two sets coincides.

Assumption 7 In the wealth model, $a^i \equiv u^{i\prime}(+\infty) = 0$ or $b^i \equiv u^{i\prime}(-\infty) = +\infty$ for any i.

Corollary 12 *Consider the wealth model. Under Assumptions 6 and 7, $V^i = W^i$.*

Proof. We want to prove that $W^i \subseteq V^i$. Consider a useful portfolio $w^i \in W^i \not\subseteq V^i$, that is $\rho^i_s \equiv R^i_s w^i < 0$ for some s or, equivalently, $b^i \sum_{s \in S^i_-} \pi^i_s \rho^i_s < 0$.

$a^i = 0$ and (5) (usefulness) imply $b^i \sum_{s \in S^i_-} \pi^i_s \rho^i_s \geq 0$, that is a contradiction. Then $W^i = V^i$.

$b^i = +\infty$ implies $a^i \sum_{s \in S^i_+} \pi^i_s \rho^i_s + b^i \sum_{s \in S^i_-} \pi^i_s \rho^i_s = -\infty < 0$. According to (5), w^i is not useful, that is a contradiction. Then $W^i = V^i$. ∎

We introduce two no-arbitrage conditions for financial markets and we show their equivalence. NA0 means that a no-arbitrage price exists, that is the intersection of individual cones of no-arbitrage prices is non-empty. NA1 means that an asset price system exists such that any portfolio yielding a positive return in some state and non-negative returns in the others, violates the financial budget constraint. In both the cases, individuals renounce to speculate. Formal definitions are now supplied.

A vector q is a no-arbitrage asset price system for agent i if $qw^i > 0$ for any $w^i \in W^i \setminus \{0\}$.

Condition 13 *(NA0) There exists a vector q that is a no-arbitrage asset price system for any agent i.*

Let S^i denote the cone of no-arbitrage prices for agent i. The following corollary characterizes NA0.

Corollary 14 *NA0 is equivalent to $\cap_i S^i \neq \varnothing$.*

Proof. We observe that $S^i = -\operatorname{int}(W^i)^0$ where $(W^i)^0$ is the polar[6] of W^i. Under Assumption 3, the sets W^i do not contain lines and the sets S^i are nonempty (see Dana, Le Van and Magnien [5] among others). Let $\cap_i S^i$ be the intersection of all the cones of no-arbitrage prices. Thus, if NA0 holds, q belongs to $\cap_i S^i$, and, if $\cap_i S^i$ is nonempty, NA0 holds. ∎

Another no-arbitrage concept can be found in the financial literature.

Condition 15 *(NA1) There exists a vector q such that, for any agent i and any portfolio $x^i \in \mathbb{R}^K$ with $R^i_s x^i \geq 0$ for any s and $R^i_t x^i > 0$ for some t, we have $qx^i > 0$.*

---

[6]The polar cone of a set $X \subseteq \mathbb{R}^K$ is defined as $X^0 \equiv \{y \in \mathbb{R}^K : y^T x \leq 0 \, for \, any \, x \in X\}$.

Conditions NA0 and NA1 are similar and turn out to be equivalent under additional assumptions.

Proposition 16 *Let Assumption 6 hold.*
(1) NA0 implies NA1.
(2) Consider the consumption model or, under Assumption 7, the wealth model. Under Assumption 3, NA0 is equivalent to NA1.

Proof. (1) Focus on agent i. If NA0 holds, $qw^i > 0$ for any $w^i \in W^i \setminus \{0\}$. For any portfolio $w^i \in \mathbb{R}^K$ with $R_s^i w^i \geq 0$ for any s and $R_t^i w^i > 0$ for some t, we have $w^i \in V^i \setminus \{0\} \subseteq W^i \setminus \{0\}$ from Proposition 9. Thus, $qw^i > 0$. This argument applies whatever the agent i. Thus, NA1 holds.

(2) We know from point (1) that NA0 implies NA1 in general. We want to prove that NA1 implies also NA0 in the consumption model or in the wealth model with the additional restrictions of point (2). Focus on agent i. From Proposition 9 and Corollary 12, we have $V^i = W^i$. Let $w^i \in W^i \setminus \{0\}$. Then, $w^i \in V^i \setminus \{0\}$ and $R_s^i w^i \geq 0$ for any s. Moreover, $R_t^i w^i > 0$ for some t because of Assumption 3. Under NA1, we obtain $qw^i > 0$, that is q is a no-arbitrage price for agent i. This argument applies whatever the agent i. Thus, NA0 is satisfied. ∎

4 Equilibrium

In this section, we prove the existence of equilibrium in economies with international asset and good markets. We eventually provide an example of nonexistence when UIRP fails.

In a Walrasian equilibrium, price-taker agents diversify their portfolios and smooth consumption across the states of nature according to their preferences, while asset and good markets clear. The proof of existence requires a formal definition.

4.1 Definitions

The usual definition of equilibrium can be adapted to take in account the financial and international aspects of the model. We introduce the sets of wealth allocations Y^i and consumption allocations Y_+^i generated by the purchase of a portfolio x^i of financial assets:

$$Y^i \equiv \{c^i \in \mathbb{R}^S : there \, is \, x^i \in \mathbb{R}^K \, such \, that \, c^i = e^i + R^i x^i\}$$
$$Y_+^i \equiv \{c^i \in \mathbb{R}^S : there \, is \, x^i \in \mathbb{R}^K \, such \, that \, c_s^i = e_s^i + R_s^i x^i \geq 0 \, for \, any \, s\}$$

Definition 17 *(equilibrium) Given beliefs and endowments (π, e), and returns and preferences (R^i, u^i) for any country i, an equilibrium is a list of prices and allocations $(p, q, c, x)^*$, with $q^* \neq 0$, such that: individual plans are optimal (points (1) and (2) below) and markets clear (points (3) and (4)):*

(1) portfolios are optimized given the asset prices q^*: for any agent i, x^{i*} solves program (3);

(2) for any agent i, $c^{i*} = e^i + R^i x^{i*}$ solves the program

$$\max \sum_s \pi_s^i u^i \left(c_s^i \right)$$

$$p^{i*} c^i \leq p^{i*} e^i$$

with $c^i \in Y^i$ in the case of wealth model or $c^i \in Y_+^i$ in the case of consumption model;

(3) asset markets clear: $\sum_i x^{i*} = 0$;

(4) the trade balance is satisfied in any state of nature s: $\sum_i p_s^{i*} c_s^{i*} = \sum_i p_s^{i*} e_s^i$.

Condition (2) implies $p^{i*} R^i x^{i*} = 0$: in equilibrium, the average value across the states of any portfolio is zero.

Consider conditions (2) and (4) together. Condition (4) is the balance in the consumption good world market when the state is s. Multiplying the sth row $c_s^{i*} = e_s^i + R_s^i x^{i*}$ in (2) by p^{i*}, summing over i and comparing with (4), we get $\sum_i p_s^{i*} R_s^i x^{i*} = 0$: in period 1, the value of the returns obtained from asset trade of period 0, is zero.

The definition of equilibrium with transfer weakens the notion of equilibrium.

Definition 18 *(equilibrium with transfer)* $(p, q, c, x)^*$ *is an equilibrium with transfer if $c^{i*} = e^i + R^i x^{i*}$ for any i and*

(1) $\sum_s \pi_s^i u^i \left(e_s^i + R_s^i y^i \right) > \sum_s \pi_s^i u^i \left(e_s^i + R_s^i x^{i*} \right)$ implies $q^* y^i > q^* x^{i*}$ for any i,

(2) $\sum_s \pi_s^i u^i \left(e_s^i + R_s^i y^i \right) > \sum_s \pi_s^i u^i \left(e_s^i + R_s^i x^{i*} \right)$ implies $p^{i*} R^i y^i > p^{i*} R^i x^{i*}$,

(3) $\sum_i x^{i*} = 0$,

(4) $\sum_i p_s^{i*} c_s^{i*} = \sum_i p_s^{i*} e_s^i$ for any s.

We observe that this definition does not require the individual budget constraints $p^{*i} c^{*i} = p^{*i} e^i$ to be satisfied. We will apply this equilibrium concept in the case of risk neutrality at the end of the paper jointly with the notion of Pareto allocation.

Definition 19 *(Pareto) A portfolio matrix x is a Pareto allocation if it is a net trade and there exists no other net trade y which satisfies:*

(1) $\sum_s \pi_s^i u^i \left(e_s^i + R_s^i y^i \right) \geq \sum_s \pi_s^i u^i \left(e_s^i + R_s^i x^i \right)$ for any i,

(2) $\sum_s \pi_s^j u^j \left(e_s^j + R_s^j y^j \right) > \sum_s \pi_s^j u^j \left(e_s^j + R_s^j x^j \right)$ for some j.

Eventually, notice that an equilibrium in the international asset markets jointly with a disequilibrium in the international good markets is possible when the exchange rates are of no use, because any agent values consumption in the currency of her country and, thus, she leaves aside these exchange rates to compute her consumption demand.

4.2 Properties

A general equilibrium price system excludes financial arbitrage opportunities. One may question whether no-arbitrage conditions are only necessary or also sufficient for equilibrium existence. In this section, we show that the equilibrium asset prices satisfy the no-arbitrage conditions NA0 and NA1. In the successive section, after the proof of equilibrium existence, a characterization in terms of no-arbitrage conditions and parities will be given.

Proposition 20 *(1) Under Assumptions 5 and 6, an equilibrium price q^* for the international asset markets satisfies NA1.*

(2) Let Assumption 3 also hold.

(2.1) In the consumption model, an equilibrium price q^* for the international asset markets satisfies NA0 and NA1.

(2.2) In the wealth model, if u^i is strictly concave, for any x^i and any $w^i \in W^i \setminus \{0\}$, $\sum_s \pi_s^i u^i \left(e_s^i + R_s^i \left(x^i + w^i\right)\right) > \sum_s \pi_s^i u^i \left(e_s^i + R_s^i x^i\right)$, and any equilibrium price q^* for the international asset markets satisfies NA0 and NA1.

Proof. (1) Consider the agent i and let $w^i \in \mathbb{R}^K$ satisfy $R_s^i w^i \geq 0$ for any s and $R_t^i w^i > 0$ for some t. Let $x^{i*} \in \mathbb{R}^K$ denote equilibrium portfolio of agent i. Since $\pi_s^i > 0$ for any s and u^i is strictly increasing (Assumption 5 and 6), we have $\sum_s \pi_s^i u^i \left(e_s^i + R_s^i \left(x^{i*} + w^i\right)\right) > \sum_s \pi_s^i u^i \left(e_s^i + R_s^i x^{i*}\right)$ which implies $q^* w^i > 0$. Thus, q^* verifies NA1.

(2.1) The result follows from point (1) and Proposition 16, point (2).

(2.2) From point (1), we know that NA1 holds with q^*. We want to prove that q^* verifies also NA0. Focus on agent i. Let $w^i \in W^i \setminus \{0\}$. w^i and $w^i/2$ are both useful portfolios and Definition 8, point (2), applies:

$$\sum_s \pi_s^i u^i \left(e_s^i + R_s^i \left(x^i + w^i\right)\right) \geq \sum_s \pi_s^i u^i \left(e_s^i + R_s^i x^i\right) \tag{6}$$

$$\sum_s \pi_s^i u^i \left(e_s^i + R_s^i \left(x^i + w^i\right)\right) \geq \sum_s \pi_s^i u^i \left(e_s^i + R_s^i \left(x^i + w^i/2\right)\right) \tag{7}$$

We want to prove that (6) holds with a strict inequality. The strict concavity of u^i implies:

$$\sum_s \pi_s^i u^i \left(e_s^i + R_s^i \left(x^i + \frac{w^i}{2}\right)\right)$$
$$> \frac{1}{2} \sum_s \pi_s^i u^i \left(e_s^i + R_s^i x^i\right) + \frac{1}{2} \sum_s \pi_s^i u^i \left(e_s^i + R_s^i \left(x^i + w^i\right)\right) \tag{8}$$

$w^i \neq 0$ entails $R_s^i w^i \neq 0$ for any s (Assumption 3): if (6) holds with equality, (8) becomes $\sum_s \pi_s^i u^i \left(e_s^i + R_s^i \left(x^i + w^i/2\right)\right) > \sum_s \pi_s^i u^i \left(e_s^i + R_s^i \left(x^i + w^i\right)\right)$ leading to a contradiction with (7).

Let now $(q, x)^*$ be an equilibrium for the international asset markets. For any $w^i \in W^i \setminus \{0\}$, we get $\sum_s \pi_s^i u^i \left(e_s^i + R_s^i \left(x^{i*} + w^i\right)\right) > \sum_s \pi_s^i u^i \left(e_s^i + R_s^i x^i\right)$

which implies $q^* w^i > 0$, that is q^* is a no-arbitrage price for agent i (Condition 13). This argument applies whatever the agent i. Thus, q^* satisfies NA0. ∎

We notice that $\sum_i p_s^{i*} c_s^{i*} = \sum_i p_s^{i*} e_s^i$ (market clearing in terms of value) is equivalent to $\sum_i \tau_s^{i*} c_s^{i*} = \sum_i \tau_s^{i*} e_s^{i*}$ (market clearing in terms of volume) because p^* satisfies the (absolute) PPP. Indeed, $p_s^{i*} = \tau_s^{i*} p_s^{0*}$ for any country i and any state s and, thus, the market clearing in terms of value is equivalent to $p_s^{0*} \sum_i \tau_s^{i*} c_s^{i*} = p_s^{0*} \sum_i \tau_s^{i*} e_s^i$ which is equivalent in turn to the market clearing in terms of volume.

We observe that the equivalence between NA0 and NA1 holds in the consumption model under mild assumptions and in the wealth model under the additional Assumption 7. More precisely, Proposition 16 establishes an equivalence between NA0 and NA1 in the wealth model under Assumptions 3, 6 and 7. Combining point (1) of Proposition 20 and the assumptions of Proposition 16, we also get that q^* satisfies NA0 in the wealth model.

For now, we have proved that, if an equilibrium asset price vector q^* exists, it satisfies the no-arbitrage conditions. Whether these necessary conditions are also sufficient for equilibrium existence is a natural question. However, prior to answering, a proof of equilibrium existence is given.

4.3   Existence

In this section, we prove the existence of an equilibrium with financial markets and international trade. We know also that no-arbitrage conditions are necessary properties of a general equilibrium (Proposition 20) and we address the sufficiency issue. The equilibrium existence is characterized through no-arbitrage conditions and UIRP.

Proposition 21 *Let Assumption 4 hold in the consumption model and Assumption 7 in the wealth model. Let Assumptions 1, 3, 5, and 6, and NA0 (Condition 13) hold in both the models.*

(1) There exists an equilibrium in the international asset markets $(q,x)^*$ with $q^* \gg 0$.

(2) Add furthermore UIRP. There exists an equilibrium $(p,q,c,x)^*$, with $q^* \neq 0$ (Definition 17). The consumption good prices p^{i*} satisfy the PPP, that is $p_s^{i*} = \tau_s^{i*} p_s^{0*}$ for any country i and any state s, and $q^* = p^{i*} R^i$ for any agent i.

Proof. (1) The proof is provided by Werner [18], Page and Wooders [12], Dana, Le Van, Magnien [5] among others. The strict positivity of q^* results from the strict increasingness of u^i jointly with Assumptions 1 and 3.

(2) The proof is given in the Appendix. ∎

Proposition 21 stresses the role of no-arbitrage conditions in equilibrium existence. The role of parities is now considered.

Corollary 22 *Let Assumptions 1, 3, 5 and 6, and UIRP hold.*

(1) Consider the consumption model. Under Assumption 4, there exists an equilibrium. The equilibrium prices satisfy PPP.

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(2) Consider the wealth model. Under Assumption 7, there exists an equilibrium. The equilibrium good prices satisfy the PPP and $q^* = p^{i*}R^i$ for any agent i.

Proof. Let UIRP hold. Focus on Definition 8, point (2) and notice that, under UIRP, $\tau_s^{i*}R_s^i w^i = \tau_s^{j*}R_s^j w^i$ and $R_s^i w^i \geq 0$ if and only if $R_s^j w^i \geq 0$. Thus, under UIRP, the set W^i is independent of i and hence $S^i = -\operatorname{int}(W^i)^0$ as well. To verify that the no-arbitrage condition NA0 is satisfied, it is enough to prove that S^0 is nonempty (Corollary 14).

Let $w^0 \in W^0 \setminus \{0\}$. We want to show that $R_t^0 w^0 > 0$ for some t. Assume on the contrary that $R_s^0 w^0 = 0$ for any s. Assumption 3 implies $w^0 = 0$, that is a contradiction. Now, let $q \equiv \left(\sum_s R_{s1}^0, \ldots, \sum_s R_{sK}^0\right)$. Then, $q w^0 = \sum_k w_k^0 \sum_s R_{sk}^0 = \sum_s R_s^0 w^0 > 0$ for any $w^0 \in W^0 \setminus \{0\}$, that is $q \in S^0$. Thus, $\cap_i S^i = S^0 \neq \varnothing$: according to Corollary 14, the no-arbitrage condition NA0 holds.

Point (2) of Proposition 21 applies. ∎

Consider now the portfolio matrix x and a vector of country-specific numbers $\mu \equiv \left(\mu^0, \ldots, \mu^I\right)$.

Proposition 23 *In the wealth model, let Assumptions 1, 3, 5, 6 and 7, and UIRP hold. Assume also, for any i, $a^i < u^{i\prime}\left(e_s^i + R_s^i x^i\right) < b^i$ for any x^i, where $a^i \equiv u^{i\prime}\left(+\infty\right)$ and $b^i \equiv u^{i\prime}\left(-\infty\right)$.*

An equilibrium exists if and only if a no-arbitrage price exists or, equivalently, a pair of positive numbers and portfolios (μ, x) exists such that, for any triplet (i, j, k), we have

$$\mu^{i*}\sum_s \pi_s^i u^\prime\left(e_s^i + R_s^i x^{i*}\right)R_{sk}^i = \mu^{j*}\sum_s \pi_s^j u^\prime\left(e_s^j + R_s^j x^{j*}\right)R_{sk}^j \qquad (9)$$

In addition, the system of prices p^{i*} satisfies PPP.

Proof. The no-arbitrage condition in the international asset markets is equivalent to the existence of a pair of positive numbers and portfolios (μ, x) such that for, any triplet (i, j, k), we have (9). From point (1) of Proposition 21, there exists an equilibrium in the international asset market. Under UIRP, an equilibrium exists also in the good markets (see point (2) of Corollary 22). ∎

From Proposition 21, we observe that imbalances in the international trade may come from an inappropriate system of returns. This happens when returns satisfy the no-arbitrage conditions in the asset markets (NA0 and NA1) without meeting the no-arbitrage condition in the international good markets (NAG). In this case, an equilibrium in the asset markets coexists with a disequilibrium in the good markets: an example of such a disequilibrium is provided in Subsection 4.4.

4.4   Example under risk aversion

To illustrate the necessity of UIRP to equilibrium existence in the case of risk-aversion, consider an example of nonexistence of equilibrium when UIRP fails.

UIRP means $R_s^0 = \tau_s^{i*} R_s^i$ for any i and s (Condition 2), and implies that any portfolio x^i yields the same return $R_s^0 x^i = \tau_s^{i*} R_s^i x^i$ in any country i, when valued in the currency of country 0. To highlight the nonexistence of equilibrium when UIRP fails, it is enough to focus on a minimalist exchange economy with one consumption good, two countries: $i = 0, 1$, two assets: $k = 1, 2$, and two states of nature: $s = 1, 2$. The matrices of returns on assets are given:

$$R^0 = \left(R_{sk}^0\right) \equiv \left[\begin{array}{cc} R_{11}^0 & R_{12}^0 \\ R_{21}^0 & R_{22}^0 \end{array}\right] = \left[\begin{array}{cc} 1 & 0 \\ 1 & 2 \end{array}\right] \tag{10}$$

$$R^1 = \left(R_{sk}^1\right) \equiv \left[\begin{array}{cc} R_{11}^1 & R_{12}^1 \\ R_{21}^1 & R_{22}^1 \end{array}\right] = \left[\begin{array}{cc} 0 & 1 \\ 2 & 1 \end{array}\right] \tag{11}$$

In this economy, UIRP is violated because $R_1^0 = \tau_1^{i*} R_1^i$ implies $(1,0) = \tau_1^{i*}(0,1)$, a contradiction.

Individual share the same beliefs and the states are considered equiprobable:

$$\pi \equiv \left[\begin{array}{cc} \pi_1^0 & \pi_2^0 \\ \pi_1^1 & \pi_2^1 \end{array}\right] = \frac{1}{2}\left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right]$$

The utility functions are also the same across the countries:

$$\sum_s \pi_s^i u^i\left(c_s^i\right) = \frac{1}{2}\sum_s \sqrt{c_s^i}$$

but the initial endowments differ:

$$e \equiv \left[\begin{array}{cc} e_1^0 & e_1^1 \\ e_2^0 & e_2^1 \end{array}\right] = \left[\begin{array}{cc} 1 & 1 \\ 1 & 2 \end{array}\right]$$

The sets of useful portfolios

$$W^0 = \left\{(x_1^0, x_2^0) : x_1^0 \geq 0,\ x_1^0 + 2x_2^0 \geq 0\right\}$$
$$W^1 = \left\{(x_1^1, x_2^1) : x_2^1 \geq 0,\ 2x_1^1 + x_2^1 \geq 0\right\}$$

determine the cones of no-arbitrage prices:

$$S^0 = -int\left(W^0\right)^0 = \left\{(p_1^0, p_2^0) : p_1^0 > 0,\ p_2^0 > 0,\ 2p_1^0 - p_2^0 > 0\right\}$$
$$S^1 = -int\left(W^1\right)^0 = \left\{(p_1^1, p_2^1) : p_1^1 > 0,\ p_2^1 > 0,\ 2p_2^1 - p_1^1 > 0\right\}$$

It is immediate to see that $(1,1) \in S^0 \cap S^1$. From Proposition 21, there exists a pair $(q,x)^*$ with $q^* \gg 0$ such that, for any agent i, x^{i*} is solution of program (3) and x^* is a net trade (Definition 4), that is the asset markets clear: $\sum_{i=0}^{2} x^{i*} = 0$.

Trade balances are satisfied in any state of nature s: $\sum_i p_s^{i*}\left(c_s^{i*} - e_s^i\right) = 0$ (Definition 17, point (2.2)). From equation (1), we know that $c_s^{i*} - e_s^i = R_s^i x^{i*}$. Thus, $\sum_i p_s^{i*} R_s^i x^{i*} = 0$ for any s. In our example, we get $p_1^{0*} R_{11}^0 x_1^{0*} + p_1^{0*} R_{12}^0 x_2^{0*} + p_1^{1*} R_{11}^1 x_1^{1*} + p_1^{1*} R_{12}^1 x_2^{1*} = 0$. Using (10) and (11), we find $p_1^{0*} x_1^{0*} + p_1^{1*} x_2^{1*} = 0$. Since $x_2^{0*} + x_2^{1*} = 0$, we obtain also $p_1^{0*} x_1^{0*} - p_1^{1*} x_2^{0*} = 0$, that is a

contradiction with $q_1^* x_1^{0*} + q_2^* x_2^{0*} = 0$ under price positivity if $x_1^{0*} \neq 0$. Focus then on the case $x_1^{0*} = 0$. $p_1^{0*} x_1^{0*} + p_1^{1*} x_2^{1*} = 0$ implies $x_2^{1*} = 0$. Net trade implies also $x_1^{1*} = x_2^{1*} = 0$. If this allocation is an equilibrium, it solves (3): there are positive multipliers μ^{i*} such that $q_k^* = \mu^{i*} \sum_s \pi_s^i u' \left(c_s^{i*} \right) R_{sk}^i$ for any k and $i = 0, 1$. Noticing that $c_s^{i*} = e_s^i + R_{s1}^i x_1^{i*} + R_{s2}^i x_2^{i*}$ and $\left(c_1^0, c_2^0, c_1^1, c_2^1 \right)^* = (1, 1, 1, 2)$, we find

$$
q_1^* = \frac{\mu^{0*}}{2} \sum_s \frac{\pi_s^0 R_{s1}^0}{\sqrt{c_s^{0*}}} = \frac{\mu^{0*}}{2} \, and q_1^* = \frac{\mu^{1*}}{2} \sum_s \frac{\pi_s^1 R_{s1}^1}{\sqrt{c_s^{1*}}} = \frac{\mu^{1*}}{2\sqrt{2}}
$$

$$
q_2^* = \frac{\mu^{0*}}{2} \sum_s \frac{\pi_s^0 R_{s2}^0}{\sqrt{c_s^{0*}}} = \frac{\mu^{0*}}{2} \, and q_2^* = \frac{\mu^{1*}}{2} \sum_s \frac{\pi_s^1 R_{s2}^1}{\sqrt{c_s^{1*}}} = \frac{\mu^{1*}}{4} \frac{1 + \sqrt{2}}{\sqrt{2}}
$$

that is impossible. This rules out any equilibrium in the good markets.

5 Risk-neutral case

Propositions 20, 21 and 23 are characterization of equilibrium existence through no-arbitrage conditions. Adding more structure in terms of fundamentals, we can give a better picture of equilibrium prices. More precisely, under the assumption of agents' risk neutrality, we can bridge prices and beliefs. In this section, not only we address the existence, but also the optimality issue. We eventually provide an example of no-trade equilibrium when UIRP fails.

5.1 Existence

In Section 4, we have proved the existence of an equilibrium under general preferences. More can be said about the equilibrium under risk-neutrality. Agents' risk-neutrality is captured by linear preferences. Before revisiting the equilibrium properties, we introduce two formal assumptions.

Assumption 8 *(risk neutrality) Preferences are linear for any agent i:* $u^i \left(c_s^i \right) = c_s^i$.

Assumptions 9 *(common beliefs) Individuals share the same beliefs:* $\pi^i = \pi^0$ *for any i.*

Propositions in the previous section bridge equilibrium existence and no-arbitrage conditions. The next Proposition and its Corollary focus on the parities and the equilibrium existence: a linear representation of equilibrium prices under risk neutrality is provided.

Consider the $(1 + I) \times S$ matrix of consumption prices p^* and a vector of state-dependent variables $z^0 \equiv \left(z_1^0, \ldots, z_S^0 \right)$.

Proposition 24 *Let Assumptions 1, 2, 5, 6 and 8 (risk neutrality), and UIRP hold.*

(1) If an equilibrium exists, there are a vector of positive numbers $\mu \in \mathbb{R}_{++}^{1+I}$ and a vector of state-dependent variables $z^0 \in \mathbb{R}^S$ such that $\mu^i \pi_s^i = \tau_s^{i*} \left(\mu^0 \pi_s^0 + z_s^0 \right)$ for any agent $i \neq 0$ and any state s, and $z^0 R^0 = 0$.

In addition, if we define the system of prices p^* with $p_s^{0*} \equiv \mu^0 \pi_s^0 + z_s^0$ for any s and $p_s^{i*} \equiv \mu^i \pi_s^i$ for any $i \neq 0$ and any s, then p^* is a system of equilibrium prices for the international good markets.

(2) Conversely, assume that there are a system of consumption good prices p^* and a vector $z^0 \in \mathbb{R}^S$ such that $p_s^{0*} \equiv \mu^0 \pi_s^0 + z_s^0$ for any state s, $p_s^{i*} \equiv \mu^i \pi_s^i = \tau_s^{i*} p_s^{0*}$ for any country $i \neq 0$ and for any state s, and $z^0 R^0 = 0$. Define also a vector of asset prices q^* such that $q^* \equiv \mu^i \pi^i R^i$ for any country i.

Then any net trade x such that, for any i, $q^* x^i = 0$ and $c^i \equiv e^i + R^i x^i$, forms with these prices an equilibrium.

Proof. See the Appendix. ∎

A corollary results immediately from Proposition 24.

Corollary 25 *Let Assumptions 1, 2, 5, 6 and 8 (risk neutrality), and UIRP hold. An equilibrium for the international asset and good markets exists if and only if agents' beliefs π admit the existence of a vector of positive numbers $\mu \in \mathbb{R}_{++}^{1+I}$ and a vector of state-dependent variables $z^0 \in \mathbb{R}^S$ such that $\mu^i \pi_s^i = \tau_s^{i*} \left(\mu^0 \pi_s^0 + z_s^0 \right)$ for any agent $i \neq 0$ and any state s, and $z^0 R^0 = 0$.*

If the return matrices are identical: $R^i = R^0$ for any i, and the asset markets are complete, we recover the well-known necessary and sufficient condition to equilibrium existence when all the agents are risk-neutral: $\pi^i = \pi^j$ for any pair i, j. Indeed, in this case, $\tau_s^{i*} = 1$ for any country i in any state s and $z^0 = 0$ since the matrix R^0 is square and invertible.

5.2 Optimality

Now, we pay attention to the optimality properties of equilibrium by connecting the notions of equilibrium with transfer and Pareto allocation. Assumptions 1 to 5 and 8 (risk neutrality) will hold in the following.

Proposition 26 *In the risk-neutral model, if there exists a vector λ^* of positive multipliers such that $\lambda^{i*} \pi^i R^i = \lambda^{0*} \pi^0 R^0$ for any i (that is all the vectors $\pi^i R^i$ are collinear), then an allocation is Pareto if, and only if, it is a net trade.*

Proof. Any Pareto allocation is a net trade, from Definition 19 of Pareto optimality. Let x be a net trade. Suppose that x is not a Pareto allocation. In this case, there exists a net trade y such that $\pi^i R^i y^i \geq \pi^i R^i x^i$ for any i and $\pi^j R^j y^j > \pi^j R^j x^j$ for some j. Thus, $\sum_i \lambda^{i*} \pi^i R^i y^i > \sum_i \lambda^{i*} \pi^i R^i x^i$, that is $0 = \lambda^{0*} \pi^0 R^0 \sum_i y^i > \lambda^{0*} \pi^0 R^0 \sum_i x^i = 0$, a contradiction. ∎

We know from the general equilibrium theory that any Pareto allocation in a closed economy is an equilibrium with transfer.

Proposition 27 *Let markets be complete in any country and assets be nonredundant.*

(1) Assume there exists a vector λ^* of positive multipliers such that $\lambda^{i*} \pi^i R^i = \lambda^{j*} \pi^j R^j$ for any i and any j, and UIRP holds. Then, if x is a net trade and

$c^i \equiv e^i + R^i x^i$ for any i, then (p^*, q^*, c, x) is an equilibrium with transfer, where $p^{i*} = \lambda^{i*} \pi^i$ for any i and $q^* = \lambda^{i*} \pi^i R^i$.

(2) Conversely, if there exists a price system $(p, q)^*$ such that, for any net trade x, (p^*, q^*, c, x) (with $c^i = e^i + R^i x^i$ for any i) is an equilibrium with transfer, then, there exists a vector λ^* of positive multipliers such that $\lambda^{i*} \pi^i R^i = \lambda^{j*} \pi^j R^j$ for any i and any j, and UIRP holds.

Proof. See the Appendix. ∎

Proposition 27 deserves some remarks.

(1) Condition $\lambda^{i*} \pi^i R^i = \lambda^{j*} \pi^j R^j$ for any i, j means that the vector $q^* = \lambda^{i*} \pi^i R^i$ satisfies NA1. Indeed, if $R^i_s x^i \geq 0$ for any s and $R^i_t x^i > 0$ for some t, then $q^* x^i = \lambda^{i*} \pi^i R^i x^i = \lambda^{i*} \sum_s \pi^i_s R^i_s x^i > 0$.

(2) Consider the wealth model. In this case, the vector q^* satisfies also the Weak No Market Arbitrage (WNMA) introduced by Werner [18]. Indeed, the set of useful portfolios for the risk-neutral model is $W^i = \left\{ w^i \in \mathbb{R}^K : \pi^i R^i w^i \geq 0 \right\}$. The set $L^i = \left\{ w^i \in \mathbb{R}^K : \pi^i R^i w^i = 0 \right\}$ is the set of vectors w^i that satisfy $\pi^i \left[e^i + R^i \left(x^i + w^i \right) \right] = \pi^i \left(e^i + R^i x^i \right)$ for any $x^i \in \mathbb{R}^K$, and is called linearity space. A vector q^* satisfies WNMA if $q^* w^i > 0$ for any $w^i \in W^i \setminus L^i$. This condition is sufficient for the existence of an equilibrium in the asset market. Here, a vector $w^i \in W_i \setminus L_i$ if and only if $\pi^i R^i w^i > 0$, that is $q^* w^i > 0$.

(3) Assumptions 1 and 5 imply $W_i \setminus L_i \neq \emptyset$ for any i. We assume that there is a vector q^* satisfying WNMA. In this case, any net trade is Pareto. Indeed, $\pi^i R^i w^i \geq 0$ implies $q^* w^i \geq 0$. If a net trade x is not a Pareto allocation, there is another net trade y such that $\pi^i R^i \left(y^i - x^i \right) \geq 0$ for any i and $\pi^j R^j \left(y^j - x^j \right) > 0$ for some j. Since q^* satisfies WNMA, $q^* \left(y^i - x^i \right) \geq 0$ for any i and $q \left(y^j - x^j \right) > 0$ for some j. Summing over i, we find a contradiction: $0 = q^* \left(\sum_i y^i - \sum_i x^i \right) = \sum_i q^* \left(y^i - x^i \right) > 0$.

(4) Assume UIRP. We can make explicit the condition $\lambda^{i*} \pi^i R^i = \lambda^{0*} \pi^0 R^0$ for any i or, equivalently, $\lambda^{i*} \sum_s R^0_s \pi^i_s / \tau^{i*}_s = \lambda^{0*} \sum_s \pi^0_s R^0_s$ for any i. Since R^0 is invertible, we get $\lambda^{i*} \pi^i_s = \tau^{*i}_s \lambda^{0*} \pi^0_s$.

(5) In the financial literature, some authors endogeneize the exchange rates by using Pareto allocations (see Dumas [6] among others). We can do the same in our risk-neutral model. However, it seems more pertinent to do that with individually rational Pareto allocations. Indeed, if the WNMA condition is satisfied by some vector q^*, $\pi^i R^i w^i \geq 0$ implies $q^* w^i \geq 0$. Hence, from the Minkowski-Farkas' Lemma, we find $q^* = \lambda^{*i} \pi^i R^i$ with $\lambda^{*i} \geq 0$ for any i. Nevertheless, since $\pi^i R^i w^i > 0$ implies $q^* w^i > 0$, actually, $\lambda_i > 0$ for any i. From Proposition 26, any net trade x^* is Pareto. x^* solves the program $\max_{x^i} \pi^i \left(e^i + R^i x^i \right)$ under the financial constraint $q^* x^i \leq q^* x^{*i}$ for any i. Let now $c^{*i} = e^i + R^i x^{*i}$ and $c^i_s = e^i_s + R^i_s x^i$ for any i. The consumption good price system p^* must satisfy $p^{*i} c^{*i} = p^{*i} e^i + q^* x^{*i}$ and c^{*i} solves the program $\max_{c^i} \pi^i c^i$ under the budget constraint $p^{*i} c^i \leq p^{*i} c^{*i}$ for any i. We then have $p^{*i} = \lambda^{*i} \pi^i$ for any i. We can define the exchange rates τ^{*i}_s between country i

and country 0 in any state s:

$$\tau_s^{*i} = \frac{p_s^{*i}}{p_s^{*0}} = \frac{\lambda^{i*}\pi_s^i}{\lambda^{0*}\pi_s^0}$$

Normalizing the price q^* with $\sum_k q_k^* = 1$, we get

$$\tau_s^{i*} = \frac{\lambda^{i*}\pi_s^i}{\lambda^{0*}\pi_s^0}\frac{\sum_k q_k^*}{\sum_k q_k^*} = \frac{\lambda^{i*}\pi_s^i}{\lambda^{0*}\pi_s^0}\frac{\sum_k \sum_t \lambda^{0*}\pi_t^0 R_{tk}^0}{\sum_k \sum_t \lambda^{i*}\pi_t^i R_{tk}^i} = \frac{\pi_s^i}{\pi_s^0}\frac{\sum_k \sum_t \pi_t^0 R_{tk}^0}{\sum_k \sum_t \pi_t^i R_{tk}^i}$$

The exchange rates are endogenously determined, but they do not depend on the net trade of assets.

If the beliefs are the same for every country, then

$$\tau_s^{i*} = \frac{\sum_k \sum_t \pi_t R_{tk}^0}{\sum_k \sum_t \pi_t R_{tk}^i} = \frac{total\,expected\,return\,in\,country\,0}{total\,expected\,return\,in\,country\,i}$$

However, in a following example, the no-trade allocation turns out to be the unique individually rational Pareto allocation which clears the international good markets under these exchange rates. In this respect, endogeneizing the exchange rates in such a way may be of little interest.

5.3 No trade

Might an equilibrium exist when UIRP fails? If a no-arbitrage condition in the asset markets holds and UIRP doesn't, an equilibrium with no trade exists in the risk-neutral case.

Proposition 28 *If UIRP does not hold and there is a vector λ^* of positive multipliers such that $\lambda^{i*}\pi^i R^i = \lambda^{j*}\pi^j R^j$ for any i and any j, then there exists a no-trade equilibrium.*

Proof. Let $p^{i*} = \lambda^{i*}\pi^i$ for any i and $q^* = \lambda^{i*}\pi^i R^i$. Let $x^* = 0$ (no trade). Hence, $c^{i*} = e^{i*}$ for any i and the following holds.
(1) $\pi^i\left(e^i + R^i y^i\right) > \pi^i\left(e^i + R^i x^{i*}\right) = \pi^i e^i$ implies $\lambda^{i*}\pi^i R^i y^i > \lambda^{i*}\pi^i R^i x^{i*} = 0$, that is $q^* y^i > q^* x^{i*} = 0$.
   (2) $p^{i*} c^{i*} = p^{i*} e^{i*}$ for any i and $\pi^i\left(e^i + R^i y^i\right) > \pi^i\left(e^i + R^i x^{i*}\right)$ implies $p^{i*} R^i y^i > p^{i*} R^i x^{i*}$.
   (3) $\sum_i x^{i*} = 0$ (no trade is net trade).
(4) $\sum_i p_s^{i*} c_s^{i*} = \sum_i p_s^{i*} e_s^{i*}$ for any s.
   Thus, points (1) to (4) of equilibrium Definition 17 are verified. ■
   We eventually observe that an individually rational net trade x with $\lambda^{i*}\pi^i R^i = \lambda^{j*}\pi^j R^j$ for any i entails $q^* x^i = 0$ for any i. Indeed, individual rationality implies $\pi^i R^i x^i \geq 0$ for any i, that is $\lambda^{i*}\pi^i R^i x^i = \lambda^{0*}\pi^0 R^0 x^i \geq 0$. Summing over i, we get $\sum_i \lambda^{i*}\pi^i R^i x^i = \lambda^{0*}\pi^0 R^0 \sum_i x^i = 0$ which gives $q^* x^i = \lambda^{i*}\pi^i R^i x^i = 0$ for any i.
   Risk neutrality (cfr. points (1) and (2) of the proof) is indispensable to Proposition 28. Under risk aversion, Proposition 28 no longer holds as illustrated in Section 4.4.

5.4 Example under risk neutrality

We consider a case where no-arbitrage conditions work in the asset, but not in the good markets, and we show that the only individually rational Pareto allocation clearing the international good markets is a no-trade allocation. More precisely, we build an economy where WNMA holds in the asset markets while UIRP does not in the good markets and we compute the exchange rates by using individually rational Pareto allocations. The example is also an illustration of Proposition 28.

Consider an exchange economy with one consumption good, two countries: $i = 0, 1$, two assets: $k = 1, 2$, and two states of nature: $s = 1, 2$. The matrices of returns on assets are given:

$$R^0 = (R^0_{sk}) \equiv \begin{bmatrix} R^0_{11} & R^0_{12} \\ R^0_{21} & R^0_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

$$R^1 = (R^1_{sk}) \equiv \begin{bmatrix} R^1_{11} & R^1_{12} \\ R^1_{21} & R^1_{22} \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 9 & 4 \end{bmatrix}$$

as well as the beliefs:

$$\pi \equiv \begin{bmatrix} \pi^0_1 & \pi^0_2 \\ \pi^1_1 & \pi^1_2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

It is immediate to check UIRP failure. Indeed, for instance, $R^0_1 = (1, 0) \neq \tau^{1*}_1 (0, 4) = \tau^{1*}_1 R^1_1$. However, the no-arbitrage condition in the asset markets holds:

$$\lambda^{0*} \left(\pi^0_1 R^0_{11} + \pi^0_2 R^0_{21} \right) = \lambda^{1*} \left(\pi^1_1 R^1_{11} + \pi^1_2 R^1_{21} \right)$$

$$\lambda^{0*} \left(\pi^0_1 R^0_{12} + \pi^0_2 R^0_{22} \right) = \lambda^{1*} \left(\pi^1_1 R^1_{12} + \pi^1_2 R^1_{22} \right)$$

with $(\lambda^0, \lambda^1)^* = (3, 1)$. Applying the formula $p^{i*}_s = \lambda^{i*} \pi^i_s$, we find the good prices:

$$p^* \equiv \begin{bmatrix} p^0_1 & p^0_2 \\ p^1_1 & p^1_2 \end{bmatrix}^* = \begin{bmatrix} 1 & 2 \\ 2/3 & 1/3 \end{bmatrix} \tag{12}$$

and the exchange rates:

$$\left(\tau^{1*}_1, \tau^{1*}_2 \right) = \left(\frac{p^{1*}_1}{p^{0*}_1}, \frac{p^{1*}_2}{p^{0*}_2} \right) = \left(\frac{2}{3}, \frac{1}{6} \right)$$

The no-trade allocation is the unique individually rational Pareto allocation which clears the international good markets under these exchange rates. To verify, focus on an individually rational net trade x, the prices p^* in (12) and

$$q^* = \lambda^{0*} \pi^0 R^0 = \lambda^{1*} \pi^1 R^1 = (3, 4)$$

Now, the trade is balanced if, and only if, $\sum_i p^{i*}_s R^i_s x^i = 0$, $s = 1, 2$. Since x is a net trade, $x^1 = -x^0$ and $\left(p^{0*}_s R^0_s - p^{1*}_s R^1_s \right) x^0 = 0$ with $s = 1, 2$, or,

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equivalently, $\left(R_s^0 - \tau_s^{1*} R_s^1\right) x^0 = 0$. Vectors $R_s^0 - \tau_s^{1*} R_s^1$ are collinear. Focus on equations $q^* x^0 = 0$ and $\left(R_1^0 - \tau_1^{1*} R_1^1\right) x^0 = 0$ (state 1) or, more explicitly, on the system $Mx^0 = 0$ with

$$M \equiv \left[\begin{array}{cc} q_1^* & q_2^* \\ R_{11}^0 - \tau_1^{1*} R_{11}^1 & R_{12}^0 - \tau_1^{1*} R_{12}^1 \end{array} \right] = \left[\begin{array}{cc} 3 & 4 \\ 1 & -8/3 \end{array} \right]$$

det $M \neq 0$ implies $x^0 = 0$ and, finally, $x^i = 0$ for any i (no trade).

6 Conclusion

Our model lies at the crossroad of three theories: asset pricing, international trade and general equilibrium. It has allowed us to address the issues of existence and optimality of a general equilibrium in a world with international asset and good markets.

The paper starts with some equivalence results: between the no-arbitrage condition in the good markets (NAG) and a parity (UIRP), and between two no-arbitrage conditions in the asset markets (NA0 and NA1).

The central proof of existence of an equilibrium in both the financial and real markets rests on an adaptation of a fixed-point argument by Dana, Le Van, Magnien [5]. As a joint product, we have characterized the equilibrium existence in terms of purchasing power and interest rate parities, and financial no-arbitrage conditions.

In order to study the optimality properties of equilibrium, we have considered the notion of equilibrium with transfer and we have revisited the welfare theorems. We know that in the case of a closed economy, any Pareto allocation is an equilibrium with transfer. But, in the case of an open economy, we need more: for instance, in the case of risk-neutrality, a Pareto allocation is an equilibrium with transfer if UIRP holds.

Modelling the coexistence of an equilibrium in the asset markets jointly with a disequilibrium in the good markets through the failure of parities remains an interesting question. Most of financial papers (Rogoff [15] among others) consider that the parities are not respected in the short run. As suggested by Frenkel and Mussa [9] in a monetary model, trade balance disequilibria seem plausible under a regime of pegged rates because relative price adjustments are achieved through slow changes in the good markets, while financial markets are mobile and integrated. This point is tackled in the paper through two examples in the case of risk aversion and risk neutrality.

7 Appendix

Proof of Proposition 21, point (2)

Let $(q, x)^*$ be an equilibrium in the international asset markets with $q^* \neq 0$. From Proposition 20, q^* is NA1. From Dana and Jeanblanc-Piqué [2], there

exists a $(1 + I) \times S$ matrix $\beta \equiv \left(\beta_s^i \right)$ such that $q^* = \sum_s \beta_s^i R_s^i$ for any i. Setting $\tilde{p}^* \equiv \beta$, we find

$$q^* = \sum_s \tilde{p}_s^{i*} R_s^i = \sum_s \frac{\tilde{p}_s^{i*}}{\tau_s^{i*}} R_s^0 = \sum_s \tilde{p}_s^{0*} R_s^0 \tag{13}$$

for any i. Let $Z \equiv \left\{ z \in \mathbb{R}^S : \sum_s z_s R_s^0 = 0 \right\}$ and observe that $Z = \{0\}$ in the case of complete asset markets $(\text{rank} R^0 = S)$.

From (13), we get, for any i, $\tilde{p}_s^{i*} = \tau_s^{i*} \left(\tilde{p}_s^{0*} + z_s^i \right)$ with $z^i \in Z$. Define $p_s^{0*} \equiv \tilde{p}_s^{0*}$ and $p_s^{i*} \equiv \tilde{p}_s^{i*} - \tau_s^{i*} z_s^i = \tau_s^{i*} \tilde{p}_s^{0*}$ for any i and any s. Let $c_s^{i*} = e_s^i + R_s^i x^{i*}$ for any i and any s. $q^* x^{i*} = 0$ implies $p^{i*} c^{i*} = p^{i*} e^i + q^* x^{i*} = p^{i*} e^i$: the budget constraint is satisfied in any country (Definition 17, point (1.2)).

Noticing that $p_s^{i*} R_s^i = p_s^{0*} R_s^0$ for any i and any s, we obtain $p_s^{i*} c_s^{i*} = p_s^{i*} e_s^i + p_s^{i*} R_s^i x^{i*} = p_s^{i*} e_s^i + p_s^{0*} R_s^0 x^{i*}$. Summing over i, we find $\sum_i p_s^{i*} c_s^{i*} = \sum_i p_s^{i*} e_s^i + p_s^{0*} R_s^0 \sum_i x^{i*} = \sum_i p_s^{i*} e_s^i$ because $\sum_i x^{i*} = 0$. Thus, the system of prices p^* satisfies the trade balance in the consumption good markets (Definition 17, point (2.2)). Clearly, this system verifies also the PPP. We notice that $q^* = \sum_s \tilde{p}_s^{i*} R_s^i = \sum_s \tau_s^{i*} \tilde{p}_s^{0*} R_s^i + \sum_s \tau_s^{i*} z_s^i R_s^i = \sum_s p_s^{i*} R_s^i + \sum_s z_s^i R_s^0 = \sum_s p_s^{i*} R_s^i$ because $z^i \in Z$.

Proof of Proposition 24

(1) Let q^* be a system of equilibrium prices for the international asset markets. Since $u^i \left(x^i \right) = x^i$ for any i, there exists a vector of positive numbers $\mu \in \mathbb{R}_{++}^{1+I}$ such that $q^* = \mu^i \sum_s \pi_s^i R_s^i$ for any i. Since $R_s^i = R_s^0 / \tau_s^{*i}$ for any s, we have $\mu^0 \sum_s \pi_s^0 R_s^0 = \mu^i \sum_s \pi_s^i R_s^0 / \tau_s^{i*}$ for any $i \neq 0$. Hence, $\mu^i \pi_s^i / \tau_s^{i*} = \mu^0 \pi_s^0 + z_s^0$ for any $i \neq 0$ and any s with $\sum_s z_s^0 R_s^0 = 0$. It is immediate to see that $\mu^i \pi_s^i = \tau_s^{i*} \left(\mu_s^0 \pi_s^0 + z_s^0 \right)$ for any $i \neq 0$ and any s. Let us define $p_s^{0*} \equiv \mu^0 \pi_s^0 + z_s^0$ for any s and $p_s^{i*} \equiv \mu^i \pi_s^i$. Thus, $p_s^{i*} = \tau_s^{i*} p_s^{0*}$ for any $i \neq 0$ and any s.

We want to check that the budget constraints are satisfied in any country under the system of prices p^*. Consider the equilibrium allocations $\left(c^i, x^i \right)^*$ in country i. We have, for any $i \neq 0$, $p^{i*} c^{i*} = p^{i*} e^i + p^{i*} R^i x^{i*} = p^{i*} e^i + \sum_s \mu^i \pi_s^i R_s^i x^{i*} = p^{i*} e^i + \mu^i \left(\pi^i R^i \right) x^{i*} = p^{i*} e^i + q^* x^{i*} = p^{i*} e^i$ since $q^* x^{i*} = 0$ at equilibrium. For $i = 0$, we obtain $p^{0*} c^{0*} = p^{0*} e^0 + \sum_s p_s^{0*} R_s^0 x^{0*} = p^{0*} e^0 + \sum_s \left(\mu^0 \pi_s^0 + z_s^0 \right) R_s^0 x^{0*} = p^{0*} e^0 + \mu^0 \pi^0 R^0 x^{0*} + z^0 R^0 x^{0*} = p^{0*} e^0 + q^* x^{0*} = p^{0*} e^0$ because $q^* x^{0*} = 0$ and $\sum_s z_s^0 R_s^0 = 0$.

We want to check now that the trade balance holds with the system of prices p^*. Multiplying $c_s^{i*} = e_s^i + R_s^i x^{i*}$ by p_s^{i*} and summing over i, we find: $\sum_i p_s^{i*} c_s^{i*} = \sum_i p_s^{i*} e_s^i + \sum_i p_s^{i*} R_s^i x^{i*} = \sum_i p_s^{i*} e_s^i + p_s^{0*} R_s^0 x^{0*} + \sum_{i \neq 0} \tau_s^{i*} p_s^{0*} R_s^i x^{i*} = \sum_i p_s^{i*} e_s^i + p_s^{0*} R_s^0 x^{0*} + \sum_{i \neq 0} p_s^{0*} R_s^0 x^{i*} = \sum_i p_s^{i*} e_s^i + p_s^{0*} R_s^0 \sum_i x^{i*} = \sum_i p_s^{i*} e_s^i$ for any s since $\sum_i x^{i*} = 0$.

(2) Let us prove the converse. First, observe that q^* is independent of i. Indeed: $q^* = \mu^i \sum_s \pi_s^i R_s^i = \sum_s p_s^{i*} R_s^i = \sum_s \tau_s^{i*} p_s^{0*} R_s^i = \sum_s p_s^{*0} R_s^0 = \mu^0 \sum_s \pi_s^0 R_s^0 + \sum_s z_s^0 R_s^0 = \mu^0 \pi^0 R^0$ since $\sum_s z_s^0 R_s^0 = 0$.

The budget constraints are satisfied. Indeed, we have, for any $i \neq 0$, $p^{i*} c^i = p^{i*} e^i + p^{i*} R^i x^i = p^{i*} e^i + q^* x^i = p^{i*} e^i$ because $q^* = p^{i*} R^i$ and $q^* x^i = 0$. For $i = 0$: $p^{0*} c^0 = p^{0*} e^0 + p^{0*} R^0 x^0 = p^{0*} e^0 + q^* x^0 = p^{0*} e^0$.

We check also the trade balance. We have, for any s: $\sum_i p_s^{i*} c_s^{i*} = \sum_i p_s^{i*} e_s^i + p_s^{0*} R_s^0 x^0 + \sum_{i \neq 0} p_s^{i*} R_s^i x^i = \sum_i p_s^{i*} e_s^i + p_s^{0*} R_s^0 x^0 + \sum_{i \neq 0} \tau_s^{i*} p_s^{0*} R_s^i x^i = \sum_i p_s^{i*} e_s^i + p_s^{0*} R_s^0 x^0 + \sum_{i \neq 0} p_s^{0*} R_s^0 x^i = \sum_i p_s^{i*} e_s^i + p_s^{0*} R_s^0 \sum_i x^i = \sum_i p_s^{i*} e_s^i$ because $\sum_i x^i = 0$ (net trade).

Let x be a portfolio net trade such that $q^* x^i = 0$ and $c^{i*} = e^i + R_s^i x^i$ for any i. Focus on a country i and consider an alternative portfolio y^i dominating x^i: $\sum_s \pi_s^i u^i \left(e_s^i + R_s^i y^i \right) > \sum_s \pi_s^i u^i \left(e_s^i + R_s^i x^i \right)$. We want to show that $q^* y^i > 0$. Equivalently, under risk-neutrality $(u^i \left(c^i \right) = c^i)$: $p_s^{0*} = \mu^0 \pi_s^0 + z_s^0$ and $p_s^{i*} = \mu^i \pi_s^i$ for $i \neq 0$. Neutrality implies $\mu^i \pi^i R^i y^i > \mu^i \pi^i R^i x^i$, that is $\left(p^{i*} - z^i \right) R^i y^i > \left(p^{*i} - z^i \right) R^i x^i$ with $z_s^i = 0$ when $i \neq 0$. Therefore, $p^{i*} R^i y^i > p^{i*} R^i x^i$ because $z^i R^i = 0$, and, eventually, $q^* y^i > q^* x^i = 0$ because $q^* = p^{i*} R^i$. Thus, $\left(q^*, x^i \right)$ solves (3).

Proof of Proposition 27

(1) Let x be a net trade and $c^i = e^i + R^i x^i$ for any i. Then:

(1.1) $\pi^i \left(e^i + R^i y^i \right) > \pi^i \left(e^i + R^i x^i \right)$ implies $\lambda^{i*} \pi^i R^i y^i > \lambda^{i*} \pi^i R^i x^i$, that is $q^* y^i > q^* x^i$,

(1.2) $\pi^i \left(e^i + R^i y^i \right) > \pi^i \left(e^i + R^i x^i \right)$ implies $\lambda^{i*} \pi^i R^i y^i > \lambda^{i*} \pi^i R^i x^i$, that is $p^i R^i y^i > p^i R^i x^i$,

(1.3) $\sum_i x^i = 0$ since x is a net trade.

(1.4) In addition, $q^* = p^{i*} R^i = \sum_s R_s^0 p_s^{i*} / \tau_s^{i*}$ and $q^* = p^{0*} R^0$. Thus, $p_s^{0*} = p_s^{i*} / \tau_s^{i*}$ for any s because asset markets are complete and nonredundant and, thus, R^0 is regular. Therefore, for any s:

$$
\begin{aligned}
\sum_i p_s^{i*} c_s^i &= \sum_i p_s^{i*} \left(e_s^i + R_s^i x^i \right) = \sum_i p_s^{i*} e_s^i + \sum_i p_s^{0*} R_s^0 x^i \\
&= \sum_i p_s^{i*} e_s^i + p_s^{0*} R_s^0 \sum_i x^i = \sum_i p_s^{*i} e_s^i
\end{aligned}
$$

Points (1.1) to (1.4) prove that (p^*, q^*, c, x) is an equilibrium with transfer.

(2) Conversely, let x be a net trade and (p^*, q^*, c, x) be the associated equilibrium with transfer.

First, $\pi^i R^i y^i > \pi^i R^i x^i$ implies $q^* y^i > q^* x^i$ for any i (Definition 18, point (1)) and, hence, $\pi^i R^i y^i \geq \pi^i R^i x^i$ implies $q^* y^i \geq q^* x^i$ for any i. Thus, $\pi^i R^i z^i \leq 0$ for any i entails $q^* z^i \leq 0$, where $z^i \equiv x^i - y^i$ for any i. The Minkowski-Farkas' Lemma[7] applies and we obtain the existence of a vector λ^* of nonnegative multipliers such that $q^* = \lambda^{i*} \pi^i R^i$ for any i (for a proof, see Florenzano and Le Van [8], page 31). However, since $\pi^i R^i z^i < 0$ for any i implies $q^* z^i < 0$, we get $\lambda_i > 0$ for any i.

Second, we have $\sum_i p_s^{i*} c_s^i = \sum_i p_s^{i*} e_s^i$ or, equivalently, $\sum_i p_s^{i*} R_s^i x^i = 0$ for any s. In other words, $\sum_i x^i = 0$ implies $\sum_i p_s^{i*} R_s^i x^i = 0$ for any s.

Consider a matrix A with rows a_k. Assume that $Az = 0$ implies $bz = 0$, where b is a row, while z is a column of the same dimension. We claim that $b = \sum_k \mu_k a_k$ with $\mu_k \in \mathbb{R}$. Indeed, we observe that $(Az = 0 \Rightarrow bz = 0)$

---

[7]The implication: $a_k x \leq 0$ for any $k \Rightarrow bx \leq 0$, is equivalent to the existence of nonnegative multipliers μ_k such that $b = \sum_k \mu_k a_k$.

is equivalent to ($-a_k z \leq 0$ and $a_k z \leq 0$ for any $k \Rightarrow bz = 0$). Thus, the Minkowski-Farkas' Lemma applies

$$b = \sum_k \mu_k^- (-a_k) + \sum_k \mu_k^+ a_k = \sum_k \mu_k a_k \qquad (14)$$

with $\mu_k \equiv \mu_k^+ - \mu_k^-$ (notice that the multipliers μ_k are not required to be nonnegative).

Let now $\mathbf{0} \equiv (0, \dots, 0) \in \mathbb{R}^{1+I}$, $\mathbf{1} \equiv (1, \dots, 1) \in \mathbb{R}^{1+I}$ and

$$A \equiv \begin{bmatrix} \mathbf{1} & \cdots & \mathbf{0} \\ \vdots & & \vdots \\ \mathbf{0} & \cdots & \mathbf{1} \end{bmatrix}$$

(notice that, in general, A is rectangular). Let $z \equiv vec\left(x^T\right)$ where x is a $K \times (1+I)$ matrix. x is a net trade iff $Az = 0$. Let $\mathbf{b}_{sk} \equiv \left(p_s^{0*} R_{sk}^0, \dots, p_s^{I*} R_{sk}^I\right) \in \mathbb{R}^{1+I}$ and $b_s \equiv (\mathbf{b}_{s1}, \dots, \mathbf{b}_{sK})$. ($\sum_i x^i = 0 \Rightarrow \sum_i p_s^{i*} R_s^i x^i = 0$ for any s) is equivalent to ($Az = 0 \Rightarrow b_s z = 0$). In this case, expression (14) works and we obtain $b_s = \sum_k \mu_{sk}^* a_k$, that is $\mathbf{b}_{sk} = (\mu_{sk}^*, \dots, \mu_{sk}^*)$ or, more explicitly, $p_s^{i*} R_{sk}^i = p_s^{0*} R_{sk}^0 = \mu_{sk}^*$ for any i and $R_s^0 = \tau_s^{i*} R_s^i$ with $\tau_s^{i*} = p_s^{i*}/p_s^{0*} > 0$, that is UIRP. The proof is now complete.

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