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# On existence, efficiency and bubbles of Ramsey equilibrium with borrowing constraints*

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**Abstract**

We address the fundamental issues of existence and efficiency of an equilibrium in a Ramsey model with many agents, where agents have heterogenous discounting, elastic labor supply and face borrowing constraints. The existence of rational bubbles is also tackled. In the first part, we prove the equilibrium existence in a truncated bounded economy through a fixed-point argument by Gale and Mas-Colell (1975). This equilibrium is also an equilibrium of any unbounded economy with the same fundamentals. The proof of existence is eventually given for an infinite-horizon economy as a limit of a sequence of truncated economies. Our general approach is suitable for applications to other models with different market imperfections. In the second part, we show the impossibility of bubbles in a productive economy and we give sufficient conditions for equilibrium efficiency.

*Keywords:* Ramsey equilibrium, existence, efficiency, bubbles, heterogeneous agents, endogenous labor supply, borrowing constraint.

*JEL classification:* C62, D31, D91, G10.

## 1 Introduction

Frank Ramsey's (1928) seminal article on optimal capital accumulation is the fundamental model underlying much contemporary research in macroeconomic

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dynamics and growth theory. Although researchers modify the basic optimum growth model in many ways, it is common to assume an infinitely-lived representative agent whose preferences coincide with a central planner in economies for which the fundamental welfare theorems hold. Infinitely-lived representative agents also appear in models with various distortions even though a planning interpretation is no longer possible. Ramsey also formulated a conjecture about the long-run for an economy with several infinitely-lived agents, each maximizing a discounted sum of future utilities. His conjecture concerned the case in which each agent's pure rate of time preference differed. That is, some agents were more patient than others. His formal conjecture stated: "*... equilibrium would be attained by a division into two classes, the thrifty enjoying bliss and the improvident at the subsistence level*" (Ramsey (1928), p. 559). This sentence ends the paper and means that, in the long run, the most patient agent(s) would hold all the capital, while the others would consume at the minimum level necessary to sustain their lives. Ramsey did not spell out the details of the equilibrium model that might give rise to this long-run solution, nor did he offer a notion of equilibrium for such an economy.

Becker (1980) proposed such a theory and demonstrated the Ramsey conjecture for that framework. Becker's model replaced subsistence by having workers earn a wage from the provision of their labor. A critical component of his model economy was a restriction placed on how much borrowing agents could undertake. This borrowing constraint binds for all but the most patient agent in the steady state solution. If this constraint had not been imposed, then a steady state might not even exist: for example, starting at Becker's steady state allocation, the relatively impatient households would borrow against the future stream of their labor incomes, consume more in the present and accept their consumption converges to zero as time tends to infinity even if rental and wage prices adjust. Thus, a binding borrowing constraint is critical for the existence of a steady state and it results in positive consumption equal to wage income at each time in the steady state for each agent more impatient than the most patient one. The agent with the lowest rate of time preference, or equivalently, the largest utility discount factor is defined as the most patient household. In the borrowing constrained steady state, that agent consumes a rental income from owning the economy's capital stock as well as its labor income. A complicating factor in dynamic analysis with this form of budget constraint occurs whenever an agent has zero capital at some time while also earning a positive wage income. Each agent always has the option to save and earn rental income in addition to labor income in the next period even if the agent has no capital. The relatively impatient agents rationally choose not to exercise this option in the steady state equilibrium. However, this might not by so away from the steady state allocation. In fact, this option is the source of many analytical complications as well as the richness of the dynamics found in subsequent work (referenced below) by Becker with his coauthors as well as the research published by Sorger, Bosi, and Seegmuller on this problem.

The many agent Ramsey equilibrium model with a borrowing constraint has incomplete markets as some intertemporal trades are excluded from the agents' opportunity sets by ruling out borrowing against the discounted value of future wage income. Models without this constraint permit all technically feasible intertemporal trades and are said to exhibit complete markets. The complete market case has been examined by many authors. The papers by Le Van and Vailakis (2003), Le Van et al. (2007), and Becker (2012) model complete markets and confirm the relatively impatient agents consumption tends to zero in the long-run within one-sector models. Le Van et al. (2007), which is pertinent for our paper, show this holds even when there is a labor-leisure choice. Earlier work along those lines in multi-sector models appears in Bewley (1982), Coles (1985, 1986), and for an exchange economy in Rader (1971, 1972, 1981) and Kehoe (1985).

The focus of our paper concerns the borrowing constrained Ramsey equilibrium model first present in Becker (1980) and developed over nearly thirty years in a series of papers included in our bibliography along with a survey paper published in 2006. Borrowing constraints constitute a credit market imperfection that has strong implications for the basic properties equilibria enjoy compared to the complete market case. In particular, there are three broad arenas in which the two models solutions differ in critical ways. The first concerns optimality. Credit market incompleteness entails the failure of the first welfare theorem. Even more strongly, it is now known from an example constructed by Becker et al. (2011) that a Ramsey equilibrium may be inefficient in the sense of Malinvaud (1953). As a matter of fact, it is no longer possible to prove the existence of a competitive equilibrium by studying the set of Pareto-efficient allocations using the Negishi (1960) approach as done by Le Van and Vailakis (2003) and Le Van et al. (2007), among others, in the absence of credit market imperfections. Also, see Kehoe et al. (1989, 1990) for related uses of the Negishi approach in many agent Ramsey models (with each agent possessing the same discount factor).

The second difference, already noted above, concerns stationary equilibrium properties. Impatient agents consume in the borrowing constrained stationary equilibrium. However, when these constraints are relaxed the steady state vanishes as shown in the complete markets counterpart by Le Van and Vailakis (2003) and Le Van et al. (2007). In fact, they show the equilibrium aggregate capital sequence converges to a particular capital stock, but this stock is not itself a steady state stock.

The third difference concerns dynamics of the aggregate equilibrium capital stock sequence. Representative agent optimal growth – perfect foresight equilibrium models generate optimal/equilibrium capital sequences which are monotonically convergent to a modified Golden-Rule steady state. This changes in presence of borrowing constraints. Persistent cycles can arise (Becker and Foias (1987, 1994), Sorger (1994)). Indeed, there can be chaotic solutions as well (Sorger (1995)). The source of these fluctuations lies in the possibility that aggregate capital income monotonicity can fail when the elasticity of substitution in the production functions is less than one. By way of contrast, when production is Cobb-Douglas, Becker and Foias (1987) show that equilibrium aggregate capital sequence is eventually monotonic and converges to the steady state capital stock.

The Ramsey conjecture holds under perfect competition with a borrowing constraint, which is one possible market imperfection. However, other forms of imperfections make the validity of the Ramsey conjecture fragile. Prominent examples are given by distortionary taxation and market power. Sarte (1997) and Sorger (2002) study a progressive capital income taxation, while Sorger (2002, 2005, 2008) and Becker and Foias (2007) focus on the strategic interaction in the capital market. They prove the possibility of a long-run nondegenerate distribution of capital where (some) impatient agents hold capital. We focus attention in our paper entirely on the competitive case.

The closely related complete market models studied in Le Van and Vailakis (2003) and Le Van et al. (2007) also find that, under discounting heterogeneity, the monotonicity property of the representative agent counterpart does not carry over and that a twisted turnpike property holds instead (see Mitra (1979) and Becker (2012)). For example, the aggregate capital sequence can be eventually monotonic following an initial segment of damped oscillations. The very difference with the one-sector class of models à la Becker is that the optimal capital sequence always converges to a particular capital stock in the long run (which, paradoxically, is not a steady state equilibrium) and, thus, there is no room for persistent cycles.

Our paper addresses the fundamental theoretical question of the existence of an intertemporal equilibrium under borrowing constraints and heterogeneous households. Standard proofs for perfect and complete markets, such as the weighted welfare function Negishi (1960) technique, do not apply because markets are imperfect (for example, see Florenzano and Gourdel (1966) for a model with debt constraints). A different approach must be taken.

Becker et al. (1991) demonstrate the existence of an intertemporal equilibrium under borrowing constraints with inelastic labor supply. Their argument rests on the introduction of a tâtonnement map in which each fixed point yields an equilibrium. Specifically, they map the set of sequences of aggregate capital stocks to itself in such a way that a fixed point is an equilibrium. Their use of the aggregate capital stock sequences for this purpose allows them to formulate the map's domain as a compact (and convex) set in the product topology in the space of all nonnegative real sequences and thereby exploit an infinite-dimensional fixed point argument to complete their demonstration. Their proof focuses on the interaction between agents' time preference and the productivity of the underlying one-sector technology. In doing so, they admit a wide-range of preferences over future consumption streams including ones with variable rates of time preference, including some that might not even be time-consistent as is the case with the more familiar time-additive and recursive utility representations. The sequence of one-period equilibrium rental rates can be calculated at the mapping's fixed point(s). This information can be used to test whether or not the Cass (1972) criterion for inefficiency in the sense of Malinvaud (1953) holds or not in equilibrium. However, there is nothing directly in their arguments to say whether or not the equilibrium found at a fixed point is efficient. Our approach, with some additional information, tells us whether an equilibrium path is efficient.

To the best of our knowledge, Becker et al. (1991) is the sole article with a proof of equilibrium existence in an infinite discrete-time horizon with production and borrowing constraints.

In our paper, to prove the equilibrium existence, we introduce weaker assumptions governing tastes and technology than those in Becker et al. (1991). Indeed, in the case of exogenous labor supply and time-separable preferences, we only need the instantaneous utility function to be continuous and the productivity of capital at the origin to be larger than the capital depreciation rate. The method applied by Becker et al. (1991) can no longer be used under these assumptions.

The original Becker et al. (1991) existence proof does not settle the existence problem when agents have a labor-leisure choice. Many macroeconomic models, whether applied or theoretical, rest on the consumption-leisure arbitrage. This property appears in characterizations of equilibrium, that is, the properties that are satisfied if and only if an equilibrium exists. It is not enough: a characterization cannot proceed reliably without a solid foundation resolving the equilibrium existence question. Many macroeconomic simulations and policy recommendations are based on dynamic equilibrium models with endogenous labor supply and heterogenous agents, but pay no attention to equilibrium existence. In order to avoid misuses and wrong conclusions, we focus on the issue of existence in one such class of models by specializing the intertemporal preferences to be additive separable across time and as well as during each period between consumption and leisure. Our proof holds for a wide range of endowments and preferences. For example, agents can make different labor supply decisions in this separable framework.

Bosi and Seegmuller (2010) provide a local proof of existence of an intertemporal equilibrium with elastic labor supply in such a framework. Their argument rests on the existence of a local fixed point for the policy function based on the local stability properties of the steady state. Our paper supplies a considerably more general global existence proof for the elastic labor case than the local theorem reported by them. Our result is not restricted to a neighborhood of the stationary state. The steady state's stability is not the only possible equilibrium dynamic. Long-term convergence is a nice feature, but a much richer set of economies are covered by our results. For instance, our analysis is perfectly valid for a two-period cycle as long as the most patient agent's capital and consumption are bounded away from zero. We also permit capital to depreciate and thereby admit durable capital to the circulating capital framework in Becker et al. (1991).

Our setup is also suitable to address the important issue of existence of bubbles in a general equilibrium model. The seminal papers on the existence of rational bubbles in a general equilibrium context belong to the overlapping generation (OG) literature. Indeed, bubbles are more easily represented in OG economies than Ramsey models even if one questions the capacity of the OG approach to reproduce business cycles. We raise the difficult point of whether or not equilibrium bubbles can exist with many infinitely-lived agents. Market imperfections are required to support an equilibrium bubble in a Ramsey model and it is worthwhile to understand the similarities between the OG structure and the financial imperfections at work in the many agent Ramsey model. More generally, we are interested in the relation between the existence of an equilibrium bubble and efficiency in the sense of Malinvaud (1953).

The literature on rational (equilibrium) bubbles suggests strong conditions are required for their occurrence. For example, Santos and Woodford (1997) and Tirole (1982) argue that they may occur if new traders enter the market each period. Hence, the OG model appears to be a relevant dynamic framework to exhibit rational bubbles in the long run. One can refer to the seminal contribution by Samuelson (1958) for an OG endowment economy and to Tirole (1985) for an OG model with capital accumulation and rational bubbles. It is well-known from these works that there is room for a persistent bubble if the equilibrium, without a bubble, is characterized by an economic growth rate larger than the real interest rate, or equivalently, the present value of total endowments of households tends to infinity. This has something to do with our results. In our framework, there is no growth at the steady state and the real interest rate is strictly positive: this rules out any bubble.

Financially constrained economies seem to behave as OG models. In connection with the Tirole's idea that new traders should enter the market each period, more recent contributions have shown that bubbles may exist in models with heterogeneous infinite-lived households facing some borrowing constraints. Indeed, households may alternatively enter the asset market, depending whether their borrowing constraint is binding or not (see Kocherlakota (1992)); conditions close to those for bubbles in OG economies are required. This is also confirmed in our setup where a productive asset is considered.

Finally, it is worthwhile to note that a recent line of research studies the existence of rational bubbles and their economic role when entrepreneurs face financial constraints (Kocherlakota (2009), Martin and Ventura (2011), Fahri and Tirole (2012)). Alternatively, we focus on the case where households are the borrowing constrained agents.

In our paper, the original approach to the equilibrium existence in economies with elastic labor supply is complemented by two other important results. First, our proof leads us to study the occurrence of asset price bubbles under agents' heterogeneity and market imperfections. Here, the present-value price of a unit of capital at each time defines a sequence of asset prices in our model. Second, we address whether or not the equilibrium is efficient by the Malinvaud (1953) criterion. As indicated above, one resolution of the efficiency question occurs when an additional property obtains that is analogous to one assumed by Becker and Mitra (2011) in the inelastic labor supply case. The way in which we rule out bubbly equilibria leads also links to another way to verify efficiency in Propositions 2 and 3.

The fact that we can exclude asset pricing bubbles, at least in some important cases, is an interesting consequence of our resolution of the existence problem for this class of production economies with credit constraints. For instance, it is known that in models with only financial assets, there is room for asset price bubbles (see for instance Le Van et al. (2011)). The novelty of our paper is that, under very mild assumptions, the introduction of a productive sector rules out the possibility of bubbles. In other words, investing in physical assets in an economy with borrowing constraints, but without money, can be one way of avoiding the emergence of rational speculative bubbles.

Becker et al. (1991) use infinite-dimensional commodity space techniques to prove an equilibrium exists. We solve the existence problem by taking a different route altogether. We begin by establishing an equilibrium exists for each possible finite-horizon economy. Florenzano (1999) provided a proof of equilibrium existence in a two-period incomplete market model by considering budget set correspondences and an auctioneer's correspondence, and using the Gale-Mas-Colell's (1975) fixed-point theorem. We also introduce the representative firm's correspondence and we apply the same idea to a truncated economy. Each of these economies, parameterized by the length of the horizon, is an approximation to the infinite horizon economy. However, infinite-dimensional problems have a different set of difficulties than finite dimensional models. By passing to the limit as time goes to infinity we find limiting sequences of prices and quantities which are verified to meet our equilibrium conditions. More precisely, the existence proof takes three steps:

- We first consider a time-truncated economy. Since the feasible allocations sets of our economy are uniformly bounded, we prove that there exists an equilibrium in a time-truncated bounded economy by Gale and Mas-Colell's (1975) theorem. Actually, this equilibrium turns out to be an equilibrium for the time-truncated economy as the uniform bounds are relaxed, as is commonly shown in finite-dimensional commodity space general equilibrium existence proofs.
- Second, we take the limit of a sequence of truncated, unbounded economies, and prove the existence of an intertemporal equilibrium in the limit economy.
- Third, we define an asset price bubble, provide conditions for a bubbleless equilibrium and conditions for equilibrium efficiency.

Most of the formal proofs are given in Appendices 1 to 3.

## 2 The Ramsey model with heterogeneous households and endogenous labor

We present the basic framework we use. Time is discrete and the economy is perfectly competitive. There are two types of agents in the economy, firms and heterogeneous infinitely-lived households facing borrowing constraints. We clarify now the assumptions and the behavior of these two types of agents.

## 2.1 Firms

A representative firm produces a unique final good. The technology is represented by a constant returns to scale production function: $F(K_t, L_t)$, where $K_t$ and $L_t$ denote the demands for capital and labor. Profit maximization is correctly defined under the following assumption.

**Assumption 1** *The production function $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ is $C^1$, homogeneous of degree one, strictly increasing and strictly concave. Inputs are essential: $F(0, L) = F(K, 0) = 0$. Limit conditions for production and productivity hold: $F(K, L) \to +\infty$ either when $L > 0$ and $K \to +\infty$ or when $K > 0$ and $L \to +\infty$; in addition, $F_L(K, 0) = +\infty$ for any $K > 0$.*

The price of the final good, the return on capital and the wage rate are denoted by $p_t$, $r_t$ and $w_t$ respectively. Profit maximization:

$$\max_{K_t, L_t} \left[p_t F(K_t, L_t) - r_t K_t - w_t L_t\right]$$

gives $\partial F/\partial K_t = r_t/p_t$ and $\partial F/\partial L_t = w_t/p_t$.

To simplify the proof of equilibrium existence, we introduce boundary conditions on capital productivity when the labor supply is maximal and equal to $m$.

**Assumption 2** $(\partial F/\partial K_t)(0, m) > \delta$ *and* $(\partial F/\partial K_t)(+\infty, m) < \delta$, *where* $\delta \in (0, 1)$ *denotes the rate of capital depreciation.*

## 2.2 Households

We consider an economy without population growth where $m$ households work and consume. Each household $i$ is endowed with $k_{i0}$ units of capital at period 0 and one unit of time per period. Leisure demand of agent $i$ at time $t$ is denoted by $\lambda_{it}$ and the individual labor supply is given by $l_{it} = 1 - \lambda_{it}$. Individual wealth and consumption demand at time $t$ are denoted by $k_{it}$ and $c_{it}$.

Initial capital endowments are supposed to be positive.

**Assumption 3** $k_{i0} > 0$ *for* $i = 1, \ldots, m$.

Each household maximizes a utility separable over time: $\sum_{t=0}^{T} \beta_i^t u_i(c_{it}, \lambda_{it})$, where $\beta_i \in (0, 1)$ is the discount factor of agent $i$.

**Assumption 4** $u_i : \mathbb{R}_+^2 \to \mathbb{R}$ *is $C^0$, strictly increasing and concave.*

We observe a threefold heterogeneity: in terms of endowments $(k_{i0})$, discounting $(\beta_i)$ and per-period utility $(u_i)$.

In any period, the household faces a budget constraint:

$$p_t \left[c_{it} + k_{it+1} - (1 - \delta) k_{it}\right] \leq r_t k_{it} + w_t (1 - \lambda_{it})$$

It is known that, in economies with heterogenous discounting and no borrowing constraints, impatient agents borrow, consume more and work less in the short run, and that they consume less and work more in the long run to refund the debt to patient agents (see Le Van et al. (2007)). In our model, as in Becker (1980), agents are prevented from borrowing: $k_{it} \geq 0$ for $i = 1, \ldots, m$ and $t = 1, 2, \ldots$

# 3 Definition of equilibrium

We define infinite-horizon sequences of prices and quantities:

$$\left(\mathbf{p}, \mathbf{r}, \mathbf{w}, (\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i)_{i=1}^m, \mathbf{K}, \mathbf{L}\right)$$

where

$$\begin{aligned}
(\mathbf{p}, \mathbf{r}, \mathbf{w}) &\equiv \left((p_t)_{t=0}^\infty, (r_t)_{t=0}^\infty, (w_t)_{t=0}^\infty\right) \in \mathbb{R}^\infty \times \mathbb{R}_+^\infty \times \mathbb{R}_+^\infty \\
(\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i) &\equiv \left((c_{it})_{t=0}^\infty, (k_{it})_{t=1}^\infty, (\lambda_{it})_{t=0}^\infty\right) \in \mathbb{R}_+^\infty \times \mathbb{R}^\infty \times \mathbb{R}_+^\infty \\
(\mathbf{K}, \mathbf{L}) &\equiv \left((K_t)_{t=0}^\infty, (L_t)_{t=0}^\infty\right) \in \mathbb{R}_+^\infty \times \mathbb{R}_+^\infty
\end{aligned}$$

with $i = 1, \ldots, m$.

**Definition 1** *A Walrasian equilibrium* $\left(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}}, \left(\bar{\mathbf{c}}_i, \bar{\mathbf{k}}_i, \bar{\boldsymbol{\lambda}}_i\right)_{i=1}^m, \bar{\mathbf{K}}, \bar{\mathbf{L}}\right)$ *satisfies the following conditions.*

*(1) Price positivity:* $\bar{p}_t, \bar{r}_t, \bar{w}_t > 0$ *for* $t = 0, 1, \ldots$

*(2) Market clearing:*

$$\begin{aligned}
\textit{goods} &: \sum_{i=1}^m \left[\bar{c}_{it} + \bar{k}_{it+1} - (1-\delta)\bar{k}_{it}\right] = F\left(\bar{K}_t, \bar{L}_t\right) \\
\textit{capital} &: \bar{K}_t = \sum_{i=1}^m \bar{k}_{it} \\
\textit{labor} &: \bar{L}_t = \sum_{i=1}^m \bar{l}_{it}
\end{aligned}$$

*for* $t = 0, 1, \ldots$, *where* $l_{it} = 1 - \lambda_{it}$ *denotes the individual labor supply.*

*(3) Optimal production plans:* $\bar{p}_t F\left(\bar{K}_t, \bar{L}_t\right) - \bar{r}_t \bar{K}_t - \bar{w}_t \bar{L}_t$ *is the value of the program:* $\max\left[\bar{p}_t F(K_t, L_t) - \bar{r}_t K_t - \bar{w}_t L_t\right]$, *under the constraints* $K_t \geq 0$ *and* $L_t \geq 0$ *for* $t = 0, 1, \ldots$.

*(4) Optimal consumption plans:* $\sum_{t=0}^\infty \beta_i^t u_i\left(\bar{c}_{it}, \bar{\lambda}_{it}\right)$ *is the value of the program:* $\max \sum_{t=0}^\infty \beta_i^t u_i(c_{it}, \lambda_{it})$, *under the following constraints:*

$$\begin{aligned}
\textit{budget constraint} &: \bar{p}_t\left[c_{it} + k_{it+1} - (1-\delta)k_{it}\right] \leq \bar{r}_t k_{it} + \bar{w}_t(1 - \lambda_{it}) \\
\textit{borrowing constraint} &: k_{it+1} \geq 0 \\
\textit{leisure endowment} &: 0 \leq \lambda_{it} \leq 1 \\
\textit{capital endowment} &: k_{i0} \geq 0 \textit{ given}
\end{aligned}$$

*for* $t = 0, 1, \ldots$

The following claims are essential in our paper.

**Claim 1** *Labor supply is uniformly bounded.*

**Proof.** At the individual level, because $l_{it} = 1 - \lambda_{it} \in [0,1]$. At the aggregate level, because $0 \leq \sum_{i=1}^{m} l_{it} \leq m$. ■

The following claim is fundamental to prove the existence of an equilibrium in a finite-horizon economy. The argument mainly rests on the concavity of production function (Assumption 1) and the boundary conditions on capital productivity (Assumption 2). It ensures that capital is uniformly bounded as well as the other equilibrium components (consumption boundedness follows at the end of the section).

**Claim 2** *Under Assumptions 1 and 2, individual and aggregate capital supplies are uniformly bounded.*

**Proof.** At the individual level, because of the borrowing constraint, we have $0 \leq k_{it} \leq \sum_{h=1}^{m} k_{ht}$.

To prove that the individual capital supply is uniformly bounded, we prove that the aggregate capital supply is uniformly bounded. We want to show that $0 \leq \sum_{h=1}^{m} k_{ht} \leq \max\{x, \sum_{i=1}^{m} k_{i0}\} \equiv A$, where $x$ is the unique solution of

$$x = (1-\delta)\, x + F(x, m) \tag{1}$$

Since $F$ is $C^1$, increasing and concave, $F(0, L) = 0$ and $1-\delta+(\partial F/\partial K_t)(0, m) > 1 > 1-\delta+(\partial F/\partial K_t)(+\infty, m)$ (Assumptions 1 and 2), the nonzero solution of (1) is unique. Moreover, $x \leq y$ implies

$$(1-\delta)\, y + F(y, m) \leq y \tag{2}$$

We notice that

$$\begin{aligned}
\sum_{i=1}^{m} k_{it+1} &\leq \sum_{i=1}^{m} (c_{it} + k_{it+1}) \leq (1-\delta) \sum_{i=1}^{m} k_{it} + F\left(\sum_{i=1}^{m} k_{it}, \sum_{i=1}^{m} l_{it}\right) \\
&\leq (1-\delta) \sum_{i=1}^{m} k_{it} + F\left(\sum_{i=1}^{m} k_{it}, m\right)
\end{aligned}$$

because $F$ is increasing, the capital employed cannot exceed its aggregate supply $\sum_{i=1}^{m} k_{it}$ and $\sum_{i=1}^{m} l_{it} \leq m$. Let $x_t \equiv \sum_{i=1}^{m} k_{it}$. Then, $x_{t+1} \leq (1-\delta)\, x_t + F(x_t, m)$.

We observe that $x_0 \leq \max\{x, x_0\} \equiv A$. Therefore, $x_1 \leq (1-\delta)\, x_0 + F(x_0, m) \leq (1-\delta)\, A + F(A, m) \leq A$ because $x \leq A$ and, from (2), $(1-\delta)A + F(A, m) \leq A$. Iterating the argument, we find $x_t \leq A$ for $t = 0, 1, \ldots$ ■

**Claim 3** *Under Assumptions 1 and 2, individual and aggregate consumption demands are uniformly bounded.*

**Proof.** At the individual level, we have $0 \leq c_{it} \leq \sum_{h=1}^{m} c_{ht}$.

To prove that the individual consumption is uniformly bounded, we prove that the aggregate consumption is uniformly bounded.

$$\begin{aligned}\sum_{i=1}^{m} c_{it} &\leq \sum_{i=1}^{m} (c_{it} + k_{it+1}) \leq (1-\delta) \sum_{i=1}^{m} k_{it} + F\left(\sum_{i=1}^{m} k_{it}, m\right) \\ &\leq (1-\delta) A + F(A, m) \leq A\end{aligned}$$

■

We observe that if $A \leq y$, then $F(y, m) + (1-\delta) y \leq y$.

# 4 On the existence of equilibrium in a finite-horizon economy

We consider an economy which goes on for $T+1$ periods: $t = 0, \ldots, T$.

Focus first on a bounded economy, that is choose sufficiently large bounds for quantities:

$$\begin{aligned}X_i &\equiv \{(c_{i0}, \ldots, c_{iT}) : 0 \leq c_{it} \leq B_c\} = [0, B_c]^{T+1} \text{ with } A < B_c \\ Y_i &\equiv \{(k_{i1}, \ldots, k_{iT}) : 0 \leq k_{it} \leq B_k\} = [0, B_k]^{T} \text{ with } A < B_k \\ Z_i &\equiv \{(\lambda_{i0}, \ldots, \lambda_{iT}) : 0 \leq \lambda_{it} \leq 1\} = [0, 1]^{T+1} \\ Y &\equiv \{(K_0, \ldots, K_T) : 0 \leq K_t \leq B_K\} = [0, B_K]^{T+1} \text{ with } A < B_K < B_c \\ Z &\equiv \{(L_0, \ldots, L_T) : 0 \leq L_t \leq B_L\} = [0, B_L]^{T+1} \text{ with } m < B_L\end{aligned}$$

with $mB_k < B_K$.

We notice that $k_{i0}$ is given and that the borrowing constraints (inequalities $k_{it} \geq 0$) capture the imperfection in the credit market.[1]

Let $\mathcal{E}^T$ denote this economy with technology and preferences as in Assumptions 1 to 4. Let $X_i$, $Y_i$ and $Z_i$ be the $i$th consumer-worker's bounded sets for consumption demand, capital supply and leisure demand respectively ($i = 1, \ldots, m$). Eventually, let $Y$ and $Z$ be the firm's bounded sets for capital and labor demands.

**Proposition 1** *Under the Assumptions 1, 2, 3 and 4, there exists an equilibrium* $\left(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}}, \left(\bar{\mathbf{c}}_h, \bar{\mathbf{k}}_h, \bar{\boldsymbol{\lambda}}_h\right)_{h=1}^{m}, \bar{\mathbf{K}}, \bar{\mathbf{L}}\right)$ *for the finite-horizon bounded economy* $\mathcal{E}^T$.

**Proof.** The proof is articulated in many claims and given in Appendix 1. ■

Focus now on an unbounded economy.

**Theorem 4** *Any equilibrium of* $\mathcal{E}^T$ *is an equilibrium for the finite-horizon unbounded economy.*

[1] A possible generalization of credit constraints is $h_i \leq k_{it}$ with $h_i < 0$ given.

**Proof.** Let $\left(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}}, \left(\bar{\mathbf{c}}_h, \bar{\mathbf{k}}_h, \bar{\boldsymbol{\lambda}}_h\right)_{h=1}^m, \bar{\mathbf{K}}, \bar{\mathbf{L}}\right)$ with $\bar{p}_t, \bar{r}_t, \bar{w}_t > 0$, $t = 0, \ldots, T$, be an equilibrium of $\mathcal{E}^T$.

Let $(\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i)$ verify $\sum_{t=0}^T \beta_i^t u_i (c_{it}, \lambda_{it}) > \sum_{t=0}^T \beta_i^t u_i \left(\bar{c}_{it}, \bar{\lambda}_{it}\right)$. We want to prove that this allocation violates at least one budget constraint, that is that there exists $t$ such that

$$\bar{p}_t [c_{it} + k_{it+1} - (1-\delta) k_{it}] > \bar{r}_t k_{it} + \bar{w}_t (1 - \lambda_{it}) \tag{3}$$

Focus on a strictly convex combination of $(\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i)$ and $\left(\bar{\mathbf{c}}_i, \bar{\mathbf{k}}_i, \bar{\boldsymbol{\lambda}}_i\right)$:

$$\begin{aligned} c_{it}(\gamma) &\equiv \gamma c_{it} + (1-\gamma) \bar{c}_{it} \\ k_{it}(\gamma) &\equiv \gamma k_{it} + (1-\gamma) \bar{k}_{it} \\ \lambda_{it}(\gamma) &\equiv \gamma \lambda_{it} + (1-\gamma) \bar{\lambda}_{it} \end{aligned} \tag{4}$$

with $0 < \gamma < 1$. Notice that we assume that the bounds satisfy $B_c, B_k, B_K > A$ and $B_L > m$ in order ensure that we enter the bounded economy when the parameter $\gamma$ is sufficiently close to 0.

Entering the bounded economy means $(\mathbf{c}_i(\gamma), \mathbf{k}_i(\gamma), \boldsymbol{\lambda}_i(\gamma)) \in X_i \times Y_i \times Z_i$. In this case, because of the concavity of the utility function, we find

$$\begin{aligned} \sum_{t=0}^T \beta_i^t u_i (c_{it}(\gamma), \lambda_{it}(\gamma)) &\geq \gamma \sum_{t=0}^T \beta_i^t u_i (c_{it}, \lambda_{it}) + (1-\gamma) \sum_{t=0}^T \beta_i^t u_i \left(\bar{c}_{it}, \bar{\lambda}_{it}\right) \\ &> \sum_{t=0}^T \beta_i^t u_i \left(\bar{c}_{it}, \bar{\lambda}_{it}\right) \end{aligned}$$

Since $(\mathbf{c}_i(\gamma), \mathbf{k}_i(\gamma), \boldsymbol{\lambda}_i(\gamma)) \in X_i \times Y_i \times Z_i$ and $\left(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}}, \left(\bar{\mathbf{c}}_h, \bar{\mathbf{k}}_h, \bar{\boldsymbol{\lambda}}_h\right)_{h=1}^m, \bar{\mathbf{K}}, \bar{\mathbf{L}}\right)$ is an equilibrium for this economy, there exists $t \in \{0, \ldots, T\}$ such that

$$\bar{p}_t [c_{it}(\gamma) + k_{it+1}(\gamma) - (1-\delta) k_{it}(\gamma)] > \bar{r}_t k_{it}(\gamma) + \bar{w}_t (1 - \lambda_{it}(\gamma))$$

Replacing (4), we get

$$\begin{aligned} & \bar{p}_t \left(\gamma c_{it} + (1-\gamma) \bar{c}_{it} + \gamma k_{it+1} + (1-\gamma) \bar{k}_{it+1} - (1-\delta) \left[\gamma k_{it} + (1-\gamma) \bar{k}_{it}\right]\right) \\ > \; & \bar{r}_t \left[\gamma k_{it} + (1-\gamma) \bar{k}_{it}\right] + \bar{w}_t \left(1 - \left[\gamma \lambda_{it} + (1-\gamma) \bar{\lambda}_{it}\right]\right) \end{aligned}$$

that is

$$\begin{aligned} & \gamma \bar{p}_t [c_{it} + k_{it+1} - (1-\delta) k_{it}] + (1-\gamma) \bar{p}_t \left[\bar{c}_{it} + \bar{k}_{it+1} - (1-\delta) \bar{k}_{it}\right] \\ > \; & \gamma [\bar{r}_t k_{it} + \bar{w}_t (1 - \lambda_{it})] + (1-\gamma) \left[\bar{r}_t \bar{k}_{it} + \bar{w}_t \left(1 - \bar{\lambda}_{it}\right)\right] \end{aligned}$$

Since $\bar{p}_t \left[\bar{c}_{it} + \bar{k}_{it+1} - (1-\delta) \bar{k}_{it}\right] = \bar{r}_t \bar{k}_{it} + \bar{w}_t \left(1 - \bar{\lambda}_{it}\right)$, we obtain (3). Thus $\left(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}}, \left(\bar{\mathbf{c}}_h, \bar{\mathbf{k}}_h, \bar{\boldsymbol{\lambda}}_h\right)_{h=1}^m, \bar{\mathbf{K}}, \bar{\mathbf{L}}\right)$ is also an equilibrium for the unbounded economy. ■

# 5 On the existence of equilibrium in an infinite-horizon economy

In the following, we need more structure and, namely, a separable utility. Let us denote by $u_i$ and $v_i$ the utilities of consumption and leisure respectively. The next assumption replaces Assumption 4.

**Assumption 5** *The utility function is separable:* $u_i(c_{it}) + v_i(\lambda_{it})$*, with* $u_i, v_i : \mathbb{R}_+ \to \mathbb{R}$ *and* $u_i, v_i \in C^1$*. In addition, we require that* $u_i(0) = v_i(0) = 0$*,* $u_i'(0) = v_i'(0) = +\infty$*,* $u_i'(c_{it}), v_i'(\lambda_{it}) > 0$ *for* $c_{it}, \lambda_{it} > 0$*, and that functions* $u, v$ *are concave.*

**Theorem 5** *Under the Assumptions 1, 2, 3 and 5, there exists an equilibrium in the infinite-horizon economy with endogenous labor supply and borrowing constraints.*

**Proof.** We consider a sequence of time-truncated economies and the associated equilibria. We prove that there exists a sequence of equilibria which converges, when the horizon $T$ goes to infinity, to an equilibrium of the infinite-horizon economy. The proof is detailed in Appendix 2. ■

# 6 Bubbles

There is considerable interest in whether or not a perfect foresight equilibrium capital asset price sequence is consistent with the notion of a rational pricing bubble. Tirole (1990), for example, argues bubbles can arise if there are multiple solutions of the forward iterates of the no-arbitrage relation for a perfect foresight equilibrium. He goes on to show the possibility that the shadow prices supporting an efficient allocation in the sense of Malinvaud (1953) can exhibit a bubble. Under some supplementary conditions, developed below and in the following section, we find a linkage between ruling out bubbles and proving efficiency of an equilibrium.

Let $\left(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}}, \left(\bar{\mathbf{c}}_i, \bar{\mathbf{k}}_i, \bar{\boldsymbol{\lambda}}_i\right)_{i=1}^m, \bar{\mathbf{K}}, \bar{\mathbf{L}}\right)$ denote an equilibrium.

**Claim 6** *For any individual* $i$*, the equilibrium sequence of multipliers* $\bar{\boldsymbol{\mu}}_i \equiv (\bar{\mu}_{it})_{t=0}^{\infty}$ *exists. The first-order conditions:*

$$\beta_i^t u_i'\left(\bar{c}_{it}^T\right) = \bar{\mu}_{it}^T \bar{p}_t^T \geq \bar{\mu}_{it+1}^T \bar{p}_{t+1}^T (1-\delta) + \bar{\mu}_{it+1}^T \bar{r}_{t+1}^T$$

*for* $i = 1, \ldots, m$*,* $t = 0, \ldots, T$*, with equality when* $\bar{k}_{it+1}^T > 0$ *(point (7) in Claim 15), are satisfied in the limit economy.*

**Proof.** The proof is given in Appendix 3.

Let us introduce a ratio of multiplier values:

$$q_{t+1} \equiv \max_i \frac{\mu_{it+1} p_{t+1}}{\mu_{it} p_t}$$

Since $\bar{K}_t > 0$ for any $t$, there exists $i$ such that $\bar{k}_{it+1} > 0$ and $\bar{\mu}_{it}\bar{p}_t = \bar{\mu}_{it+1}\left[\bar{p}_{t+1}(1-\delta) + \bar{r}_{t+1}\right]$. We observe that

$$\frac{\bar{\mu}_{it+1}\bar{p}_{t+1}}{\bar{\mu}_{it}\bar{p}_t} \leq \frac{1}{1-\delta+\bar{\rho}_{t+1}} \text{ and } \frac{\bar{\mu}_{it+1}\bar{p}_{t+1}}{\bar{\mu}_{it}\bar{p}_t} = \frac{1}{1-\delta+\bar{\rho}_{t+1}} \text{ if } \bar{k}_{it+1} > 0$$

where the real return on capital is defined by $\rho_t \equiv r_t/p_t$.

Thus, the equilibrium ratio $\bar{q}_t$ has a natural interpretation as a market discount factor:

$$\bar{q}_{t+1} = \max_i \frac{\beta_i u_i'(\bar{c}_{it+1})}{u_i'(\bar{c}_{it})} = \frac{1}{1-\delta+\bar{\rho}_{t+1}}$$

Let

$$Q_0 \equiv 1 \tag{5}$$

$$Q_t \equiv \prod_{s=1}^{t} q_s \text{ for } t > 0 \tag{6}$$

Clearly, $\bar{Q}_0 = 1$ and $\bar{Q}_t = \prod_{s=1}^{t}(1-\delta+\bar{\rho}_s)^{-1}$ for $t > 0$. $\bar{Q}_t$ is the present value of a unit of capital of period $t$. For any $t$, we obtain:

$$\bar{Q}_t = \bar{Q}_{t+1}\left(1-\delta+\bar{\rho}_{t+1}\right) \tag{7}$$

and, by induction, $1 = \bar{Q}_0 = \bar{Q}_T(1-\delta)^T + \sum_{t=1}^{T}\bar{Q}_t\bar{\rho}_t(1-\delta)^{t-1}$.

Focus on period $t$. Consider a machine (let it be one unit of capital). The machine has been purchased at the end of time $t-1$ at a price $p_{t-1}$. The price of this machine is $p_t$ at the beginning of period and $(1-\delta)p_t$ at the end of period $t$. In equilibrium, we sell $k_t$ machines at price $\tilde{p}_t \equiv (1-\delta)\bar{p}_t$ and buy $k_{t+1}$ machines at price $\bar{p}_t$. The real interest rate is expressed in terms of consumption good (equivalently in term of new capital): $\bar{\rho}_t = \bar{r}_t/\bar{p}_t$. However, the nominal interest rate can be also normalized by $(1-\delta)\bar{p}_t$:

$$\tilde{\rho}_t \equiv \frac{\bar{r}_t}{\tilde{p}_t} = \frac{\bar{\rho}_t}{1-\delta}$$

that is the amount of end-of-period machines one can buy with $\bar{r}_t$ units of numeraire.

Similarly, define $\tilde{q}_t \equiv (1-\delta)\bar{q}_t$ and $\tilde{Q}_t \equiv \prod_{s=1}^{t}\tilde{q}_s$. $\tilde{Q}_t$ is the present value of one unit of capital at the end of period $t$ after the production process with the completion of that period's activities. Thus, we get

$$\tilde{Q}_t \equiv \prod_{s=1}^{t}\left[(1-\delta)\bar{q}_t\right] = (1-\delta)^t\prod_{s=1}^{t}\bar{q}_t = \bar{Q}_t(1-\delta)^t$$

and

$$1 = \bar{Q}_T(1-\delta)^T + \sum_{t=1}^{T}\frac{\bar{\rho}_t}{1-\delta}\bar{Q}_t(1-\delta)^t = \tilde{Q}_T + \sum_{t=1}^{T}\tilde{\rho}_t\tilde{Q}_t$$

**Definition 2** *We define the fundamental value of capital as*

$$\tilde{v}_0 \equiv \sum_{t=1}^{+\infty} \tilde{\rho}_t \tilde{Q}_t \tag{8}$$

*The economy is said to experience a bubble if* $\lim_{T \to +\infty} \tilde{Q}_T > 0$. *Otherwise* ($\lim_{T \to +\infty} \tilde{Q}_T = 0$)*, there is no bubble.*

We observe that $\tilde{v}_0$ is the fundamental value of capital after the production process takes place at any time - it is formally the present discounted value of the future stream of nominal rentals discounted to time 0. In fact, we can regard this fundamental value as the discounted value to the end of any period $t$ measured in present value prices with focal date 0. If there is no equilibrium bubble, then $\tilde{v}_0 = 1$, as we would expect: one unit of capital at time zero (or, equivalently, when productive activities conclude at the end of period $t$) trades for one unit of capital in equilibrium. It is as if capital markets satisfy a form of the efficient markets model from financial economics.

Let $i = 1$ denote the most patient agent with $\beta_i < \beta_1$ for $i = 2, \ldots, m$. At the stationary equilibrium, $\bar{Q}_t$ coincides with the discount factor of the most patient agent who is also the only one with $\bar{k}_{it} > 0$ in the long run. Thus,

$$\bar{Q}_t \equiv \prod_{s=1}^{t} \bar{q}_s = \prod_{s=1}^{t} \frac{\beta_1 u_1' \left(\bar{c}_{1s+1}\right)}{u_1' \left(\bar{c}_{1s}\right)} = \prod_{s=1}^{t} \beta_1 = \beta_1^t$$

The condition for a no bubble equilibrium is clearly met in this case. Hence, we focus our attention on the crucial question of existence of equilibrium bubbles in a productive economy in a non-stationary equilibrium.

We know that, by definition, an equilibrium bubble exists if $\lim_{T \to +\infty} \tilde{Q}_T > 0$. We have

$$\tilde{Q}_T = \prod_{s=1}^{T} \frac{1-\delta}{1-\delta+\bar{\rho}_s}$$

Let us show that a productive economy experiences no bubbles in an equilibrium. The proof rests on the following lemma.

**Lemma 1** *If the economy experiences an equilibrium bubble, then* $\bar{\rho}_t$ *converges to zero.*

**Proof.** It is equivalent to prove that, if $\bar{\rho}_t$ does not converge to zero, there are no bubbles. If $\bar{\rho}_t$ does not converge, there are, equivalently, $\varepsilon > 0$ and an infinite and increasing sequence $(t_i)_{i=1}^{+\infty}$ such that $\bar{\rho}_{t_i} \geq \varepsilon$ for all $i = 1, 2, \ldots$ For $T > t_n$, we get

$$\tilde{Q}_T = \prod_{s=1}^{T} \frac{1-\delta}{1-\delta+\bar{\rho}_s} \leq \prod_{i=1}^{n} \frac{1-\delta}{1-\delta+\bar{\rho}_{t_i}} \leq \left(\frac{1-\delta}{1-\delta+\varepsilon}\right)^n$$

and

$$0 \leq \lim \sup_{T \to +\infty} \tilde{Q}_T \leq \lim_{n \to +\infty} \left( \frac{1-\delta}{1-\delta+\varepsilon} \right)^n = 0$$

■

**Lemma 2** *Let $\psi$ be a concave, continuous, strictly increasing function on $\mathbb{R}_+$, differentiable in $\mathbb{R}_{++}$. Then, $\lim_{x \to 0_+} \psi'(x) > 0$.*

**Proof.** We have always $0 < \psi'(x) \leq \lim_{x \to 0_+} \psi'(x)$, for all $x > 0$. If $\lim_{x \to 0_+} \psi'(x) = 0$, then $\psi'(x) = 0$ for every $x > 0$ against the increasingness of $\psi$. ■

Now, we are able to prove the main theorem.

**Theorem 7** *Our productive economy experiences no equilibrium bubble.*

**Proof.** First, observe that the production function $F$ satisfies Assumption 1. Second, from Lemma 2, we have

$$\lim_{b \to 0_+} F_L(1, b) > 0 \tag{9}$$

Since $F$ is homogeneous of degree one, we have, for $K > 0$ and $L > 0$, $F_K(K, L) = F_K(K/L, 1)$ and $F_L(K, L) = F_L(1, L/K)$. Let $(\bar{K}_t, \bar{L}_t)$ be an equilibrium sequence of aggregate capital stocks and labors. Observe that $\bar{\rho}_t = F_K(\bar{K}_t/\bar{L}_t, 1)$. Since $F$ is differentiable and concave, we have for any $t$

$$\bar{\rho}_t \geq \lim_{a \to +\infty} F_K(a, 1) \tag{10}$$

Suppose the economy has an equilibrium bubble, which necessarily is reflected in the equilibrium prices. Then, from Lemma 1, $\bar{\rho}_t$ converges to zero. But from (10), $\bar{K}_t/\bar{L}_t$ tends to infinity, or equivalently, $\bar{L}_t/\bar{K}_t$ goes to 0. Since $\bar{K}_t$ is positive and bounded above (see Claim 2), we obtain $\bar{L}_t \to 0$. Recall that

$$\bar{C}_t + \bar{K}_{t+1} = F(\bar{K}_t, \bar{L}_t) + (1-\delta)\bar{K}_t = \bar{K}_t \left[ F(1, \bar{L}_t/\bar{K}_t) + 1 - \delta \right]$$

and choose $\varepsilon > 0$ such that $F(1, \varepsilon) + 1 - \delta < 1$. There exists $T$ such that for any $t > T$, $K_{t+1} \leq \bar{K}_t \left[ F(1, \bar{L}_t/\bar{K}_t) + 1 - \delta \right] < [F(1, \varepsilon) + 1 - \delta] \bar{K}_t$. This implies $\bar{K}_t \to 0$ when $t$ tends to infinity, and $\bar{C}_t \to 0$ too.

Reconsider the first-order conditions of point 8 in Claim 15 (Appendix 2):

$$\bar{\omega}_t u_i'(\bar{c}_{it}) \leq v_i'(1 - \bar{l}_{it}) \tag{11}$$

We know that $\bar{\omega}_t = F_L(\bar{K}_t, \bar{L}_t)$ (see point (3) in Claim 15, Appendix 2). Since $\bar{L}_t/\bar{K}_t$ converges to 0, according to (9), we have $\lim_{t \to +\infty} \bar{\omega}_t = \lim_{t \to +\infty} F_L(1, \bar{L}_t/\bar{K}_t) > 0$. $\lim_{t \to +\infty} \bar{C}_t = 0$ implies $\lim_{t \to +\infty} \bar{c}_{it} = 0$ and $\lim_{t \to +\infty} u_i'(\bar{c}_{it}) = +\infty$ for any $i$. Thus, $\lim_{t \to +\infty} [\bar{\omega}_t u_i'(\bar{c}_{it})] = +\infty$ on the left-hand side of (11). We notice also that $\bar{L}_t \to 0$ implies $\bar{l}_{it} \to 0$ for any $i$, and, so, $\lim_{t \to +\infty} v_i'(1 - \bar{l}_{it}) = v_i'(1) < +\infty$ on the right-hand side of (11), that is a contradiction. In other terms, there are no equilibrium bubbles in our economy.

■

## 7 Efficiency

The relation between efficiency and bubbles rests on many agents and heterogeneous discounting. An inefficient equilibrium can arise when the most patient agent owns capital infinitely often but also no capital infinitely often. In this situation, it is as if agents alternate momentarily dropping out of the capital market only to reenter it later. For instance, the period three inefficient equilibrium example constructed in Becker, Dubey and Mitra (2011) based on two households exhibits one period where the most patient agent's borrowing constraint binds while the more impatient household's borrowing constraint holds for the other two periods of the three-cycle. The implied present value shadow prices satisfy the well-known Cass (1972) test for an inefficient equilibrium capital sequence. The implied present value shadow price sequence does not converge to zero. This possibility raises the prospect that an equilibrium price bubble can occur following Tirole's intuition mentioned earlier. By way of contrast, the situation where capital is eventually, and permanently, concentrated in the hands of the most patient household is crucial to obtain an efficient solution and thereby exclude equilibrium bubbles. In the case of a representative household, the capital is trivially owned forever by the only agent who is, by assumption, also the most patient agent and there is nothing to prove. The heterogeneity of agents' discount rates, combined with incomplete markets by way of borrowing constraints, creates the friction in which it is legitimate to ask whether, or not, a bubble can exist. Let us address this question by considering a specific notion of efficiency.

As above, $\left(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}}, \left(\bar{\mathbf{c}}_i, \bar{\mathbf{k}}_i, \bar{\boldsymbol{\lambda}}_i\right)_{i=1}^m, \bar{\mathbf{K}}, \bar{\mathbf{L}}\right)$ denotes an equilibrium. Following Malinvaud (1953) and Becker and Mitra (2011), we introduce the following definition.

**Definition 3** *An equilibrium is efficient if there is no sequence of total consumption, capital and labor* $(C_t, K_t, L_t)$ *which satisfies, for* $t = 0, 1, \ldots$

$$C_t + K_{t+1} - (1 - \delta) K_t = F(K_t, L_t) \tag{12}$$

*(feasibility) with* $C_t \geq \bar{C}_t$ *and* $m - L_t \geq m - \bar{L}_t$ *for* $t = 0, 1, \ldots$ *with at least one strict inequality for consumption or for leisure.* $K_0$*, the aggregate capital endowment, is given.*

**Proposition 2** *The present value of one unit of capital* $\bar{Q}_t$ *is defined by (6) and (5). If* $\lim_{t\to\infty} \bar{Q}_t = 0$*, the equilibrium is efficient.*

**Proof.** The proof is given in Appendix 3. ■

It is worthy to notice that (1) the lack of bubbles in a productive economy means $\lim_{t\to+\infty} \bar{Q}_t (1-\delta)^t = 0$ which does not imply $\lim_{t\to\infty} \bar{Q}_t = 0$, and, actually, (2) for the proof (see Appendix 3), the sufficient condition for efficiency is $\lim_{t\to\infty} \bar{Q}_t \bar{K}_{t+1} = 0$ which is the condition given by Malinvaud (1953). This condition is equivalent, in our model, to $\lim_{t\to\infty} \bar{Q}_t = 0$ since the sequence $(K_t)$ is uniformly bounded.

Becker et al. (2011) provide an example with $\delta = 1$, where the Ramsey equilibrium may be inefficient because of the lower returns on capital. Their construction assumes a two-household economy with capital depreciating completely within the period (i.e. $\delta = 1$: capital is circulating) and each agent's labor supply is perfectly inelastic. Their example's aggregate capital stock exhibits a period three cycle. Previously, Becker and Mitra (2011) showed each two-cycle equilibrium to be efficient when $\delta = 1$ and each agent's labor supply is perfectly inelastic.

The strategy behind building their inefficiency example starts with the necessary conditions for a period three cycle. Inefficiency occurs when the compound (or geometric) returns to capital investment are negative over longer and longer horizons. This occurs because the aggregate capital stock exceeds the Golden-Rule stock infinitely often. In fact, they show that the aggregate capital path can be of only one type: it attains a peak stock level above the Golden-Rule, followed by a lower capital stock also above the Golden-Rule, and then followed by a capital stock below the Golden-Rule. The implied shadow prices of capital satisfy the well-known Cass criterion (1972) for inefficiency. The detailed construction exhibits the various restrictions on the agent's utility functions and discount factors along with the needed properties on the one-sector production function that are consistent with this period three-cycle picture. Those details also track the fluctuations in individual consumption. The key to complete the inefficiency example is demonstrating the more impatient household holds all the economy's capital infinitely often while the more patient one has no capital at those times. In this way, the so-called turnpike property fails to hold. Otherwise, when the turnpike property obtains (the most patient agent has all the capital eventually and the more impatient's stocks are eventually zero and remain so thereafter), then the resulting equilibrium is known to be efficient according to Becker and Mitra's (2011) theorems. Our Proposition 3, giving a sufficient condition for efficiency, is in the same spirit as the conditions offered in Becker and Mitra's paper. The difference is that we include a labor-leisure choice option for each agent in addition to admitting durable capital ($0 < \delta < 1$). The labor-leisure choice plays a subsidiary role through the separability property in Assumption 5. The formal argument for the proposition depends on establishing a transversality condition (by Proposition 2) for capital goods prices obtains and that, in turn, only depends on there being at least one agent who eventually holds positive stocks for all time irrespective of that person's labor supply decision.

**Proposition 3** *If there are $i$ and $S$ such that, for any $t > S$, $\bar{c}_{it} \geq c > 0$ and $\bar{k}_{it} \geq k > 0$, then the equilibrium is efficient.*

**Proof.** Consider such an individual. Since $u'(\bar{c}_{it}) \leq u'(c) < +\infty$ for $t > S$, we obtain $\sum_{t=0}^{\infty} \beta_i^t u_i'(\bar{c}_{it}) < +\infty$ and, because of point (7) in Claim 15, $\sum_{t=0}^{\infty} \bar{\mu}_{it}\bar{p}_t < +\infty$. Then $\bar{\mu}_{it}\bar{p}_t \to 0$. We know that

$$\frac{\bar{\mu}_{it+1}\bar{p}_{t+1}}{\bar{\mu}_{it}\bar{p}_t} = \frac{1}{1 - \delta + \bar{\rho}_{t+1}} = \bar{q}_{t+1}$$

if $\bar{k}_{it+1} > 0$. Hence, for any $T > S$,

$$\bar{Q}_T = \bar{Q}_S \prod_{t=S}^{T-1} \bar{q}_{t+1} = \bar{Q}_S \prod_{t=S}^{T-1} \frac{\bar{\mu}_{it+1}\bar{p}_{t+1}}{\bar{\mu}_{it}\bar{p}_t} = \bar{Q}_S \frac{\bar{\mu}_{iT}\bar{p}_T}{\bar{\mu}_{iS}\bar{p}_S}$$

and, finally,

$$\lim_{T\to+\infty} \bar{Q}_T = \frac{\bar{Q}_S}{\bar{\mu}_{iS}\bar{p}_S} \lim_{T\to+\infty} (\bar{\mu}_{iT}\bar{p}_T) = 0$$

■

In our model, as in Becker (1980), nondominant consumers have positive steady state consumption. Conversely, in Le Van et al. (2007), the consumption of impatient agents asymptotically vanishes. The difference rests on the nature of the budget constraint.

Le Van et al. (2007) allow intertemporal trades of future labor for current consumption in a loan market. In other terms, each consumer's budget constraint requires total assets to be nonnegative only asymptotically. Thus, impatient agents may borrow to consume more today and, since their total assets consist only of a discounted wage income from a certain period on, they experience zero consumption asymptotically.

Conversely, to exclude agents without capital from the loan market, we require capital assets to be nonnegative at each moment of time. For impatient consumers with no capital stocks, this implies that the wage income is consumed at each time.

In Le Van et al. (2007), the First Welfare Theorem holds and the competitive equilibrium is a Pareto optimum. In our case, the possibilities for intertemporal trade are restrained and the resulting competitive equilibrium is a constrained Pareto optimum. We observe that Malinvaud efficiency is different from Pareto optimality and that Proposition 3 gives only sufficient conditions for Malinvaud efficiency.

However, as shown in Bosi and Seegmuller (2010), there exists a unique steady state with $c_i > 0$ for all $i = 1, ..., m$, $k_1 > 0$ and $k_i = 0$ for $i > 1$, where $\beta_1 > \beta_2 \geq \ldots \geq \beta_m$. This distribution of capital still holds along a dynamic path in a neighborhood of the steady state, as soon as the inequality $u_i' \left(\bar{w}_t \bar{l}_{it}\right) > \beta_i \left(1 - \delta + \bar{r}_{t+1}/\bar{p}_{t+1}\right) u_i' \left(\bar{w}_{t+1} \bar{l}_{it+1}\right)$ is satisfied for all $i > 1$. Proposition 3 in Bosi and Seegmuller (2010) provides sufficient conditions for a local convergence to a steady state with positive consumption and capital for the dominant consumer (namely, a sufficiently large elasticity of capital-labor substitution or a sufficiently large impatient agents' average elasticity of labor supply). Otherwise, a stable cycle of period two may arise around the steady state, now unstable. The cycle remains positive if the bifurcation parameter is close enough to the critical value. Under these conditions, positive thresholds for consumption and capital $(c, k > 0)$ exist and Proposition 3 applies, that is the competitive equilibrium is Malinvaud efficient.

# 8 Conclusion

In this paper, we have shown the existence of an intertemporal equilibrium with market imperfections (borrowing constraints) and addressed the issues of bubbles existence and equilibrium efficiency.

Applying the fixed-point theorem of Gale-Mas-Colell, we have proved the existence of an equilibrium in a finite-horizon bounded economy. This equilibrium turns out to be also an equilibrium of any unbounded economy with the same fundamentals. Eventually, we have shown the existence of an equilibrium in an infinite-horizon economy as a limit of a sequence of truncated economies by applying a uniform convergence argument.

The paper generalizes in one respect Becker et al. (1991) by considering an elastic labor supply, and, in another respect, Bosi and Seegmuller (2010) by providing a proof of global existence. Our methodology, inspired by Florenzano (1999), is quite general and can be applied to other Ramsey models with different market imperfections.

At the end of the paper, we have raised the question of bubbles occurrence and efficiency. In particular, we have proved that, in our productive economy, bubbles are ruled out.

# 9 Appendix 1: existence of equilibrium in a finite-horizon economy

Let us prove Proposition 1.

The idea of the proof is borrowed from Florenzano (1999).

Define a bounded price set:

$$P \equiv \{(\mathbf{p},\mathbf{r},\mathbf{w}) : -1 \leq p_t \leq 1, 0 \leq r_t \leq 1, 0 \leq w_t \leq 1, t = 0, \ldots, T\}$$

At this stage, put no restriction on the sign of $p_t$. We will prove later price positivity through an equilibrium argument.

Focus now on the budget constraints: $p_t [c_{it} + k_{it+1} - (1-\delta) k_{it}] \leq r_t k_{it} + w_t (1-\lambda_{it})$ for $t = 0, \ldots, T-1$ and $p_T [c_{iT} - (1-\delta) k_{iT}] \leq r_T k_{iT} + w_T (1-\lambda_{iT})$.

In the spirit of Bergstrom (1976), we introduce the modified budget sets:

$$\begin{aligned} & B_i(\mathbf{p},\mathbf{r},\mathbf{w}) \\ \equiv & \left\{ \begin{array}{c} (\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i) \in X_i \times Y_i \times Z_i : \\ p_t [c_{it} + k_{it+1} - (1-\delta) k_{it}] < r_t k_{it} + w_t (1-\lambda_{it}) + \gamma(p_t, r_t, w_t) \\ t = 0, \ldots, T-1 \\ p_T [c_{iT} - (1-\delta) k_{iT}] < r_T k_{iT} + w_T (1-\lambda_{iT}) + \gamma(p_T, r_T, w_T) \end{array} \right\} \\ & C_i(\mathbf{p},\mathbf{r},\mathbf{w}) \\ \equiv & \left\{ \begin{array}{c} (\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i) \in X_i \times Y_i \times Z_i : \\ p_t [c_{it} + k_{it+1} - (1-\delta) k_{it}] \leq r_t k_{it} + w_t (1-\lambda_{it}) + \gamma(p_t, r_t, w_t) \\ t = 0, \ldots, T-1 \\ p_T [c_{iT} - (1-\delta) k_{iT}] \leq r_T k_{iT} + w_T (1-\lambda_{iT}) + \gamma(p_T, r_T, w_T) \end{array} \right\} \end{aligned}$$

where $\gamma(p_t, r_t, w_t) \equiv 1 - \min\{1, |p_t| + r_t + w_t\}$. We denote by $\bar{B}_i(\mathbf{p}, \mathbf{r}, \mathbf{w})$ the closure of $B_i(\mathbf{p}, \mathbf{r}, \mathbf{w})$.

The proof of Proposition 1 goes on through Claims 8, 9 and 10 whose proofs are provided in Becker et al. (2012).

**Claim 8** *For every* $(\mathbf{p}, \mathbf{r}, \mathbf{w}) \in P$*, we have* $B_i(\mathbf{p}, \mathbf{r}, \mathbf{w}) \neq \varnothing$ *and* $C_i(\mathbf{p}, \mathbf{r}, \mathbf{w}) = \bar{B}_i(\mathbf{p}, \mathbf{r}, \mathbf{w})$.

**Claim 9** $B_i$ *is a lower semi-continuous correspondence on* $P$.

**Claim 10** $C_i$ *is upper semi-continuous on* $P$ *with compact convex values.*

In the spirit of Gale and Mas-Colell (1975, 1979), we introduce the reaction correspondences $\varphi_i(\mathbf{p}, \mathbf{r}, \mathbf{w}, (\mathbf{c}_h, \mathbf{k}_h, \boldsymbol{\lambda}_h)_{h=1}^m, \mathbf{K}, \mathbf{L})$, $i = 0, \ldots, m+1$ defined on $P \times [\times_{h=1}^m (X_h \times Y_h \times Z_h)] \times Y \times Z$, where $i = 0$ denotes an additional agent, the auctioneer, $i = 1, \ldots, m$ the consumers, and $i = m+1$ the firm. These correspondences are defined as follows.

Agent $i = 0$ (the auctioneer):

$$\begin{aligned} &\varphi_0\left(\mathbf{p}, \mathbf{r}, \mathbf{w}, (\mathbf{c}_h, \mathbf{k}_h, \boldsymbol{\lambda}_h)_{h=1}^m, \mathbf{K}, \mathbf{L}\right) \\ \equiv &\left\{ \begin{array}{c} (\tilde{\mathbf{p}}, \tilde{\mathbf{r}}, \tilde{\mathbf{w}}) \in P : \\ \sum_{t=0}^T (\tilde{p}_t - p_t)\left(\sum_{i=1}^m [c_{it} + k_{it+1} - (1-\delta) k_{it}] - F(K_t, L_t)\right) \\ + \sum_{t=0}^T (\tilde{r}_t - r_t)\left(K_t - \sum_{i=1}^m k_{it}\right) \\ + \sum_{t=0}^T (\tilde{w}_t - w_t)\left(L_t - m + \sum_{i=1}^m \lambda_{it}\right) > 0 \end{array} \right\} \end{aligned} \tag{13}$$

Agents $i = 1, \ldots, m$ (consumers-workers):

$$\begin{aligned} &\varphi_i\left(\mathbf{p}, \mathbf{r}, \mathbf{w}, (\mathbf{c}_h, \mathbf{k}_h, \boldsymbol{\lambda}_h)_{h=1}^m, \mathbf{K}, \mathbf{L}\right) \\ \equiv &\left\{ \begin{array}{c} B_i(\mathbf{p}, \mathbf{r}, \mathbf{w}) \text{ if } (\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i) \notin C_i(\mathbf{p}, \mathbf{r}, \mathbf{w}) \\ B_i(\mathbf{p}, \mathbf{r}, \mathbf{w}) \cap [P_i(\mathbf{c}_i, \boldsymbol{\lambda}_i) \times Y_i] \text{ if } (\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i) \in C_i(\mathbf{p}, \mathbf{r}, \mathbf{w}) \end{array} \right\} \end{aligned}$$

where $P_i$ is the $i$th agent's set of strictly preferred allocations: $P_i(\mathbf{c}_i, \boldsymbol{\lambda}_i) \equiv \left\{\left(\tilde{\mathbf{c}}_i, \tilde{\boldsymbol{\lambda}}_i\right) : \sum_{t=0}^T \beta_i^t u_i\left(\tilde{c}_{it}, \tilde{\lambda}_{it}\right) > \sum_{t=0}^T \beta_i^t u_i(c_{it}, \lambda_{it})\right\}$.

Agent $i = m+1$ (the firm):

$$\begin{aligned} &\varphi_{m+1}\left(\mathbf{p}, \mathbf{r}, \mathbf{w}, (\mathbf{c}_h, \mathbf{k}_h, \boldsymbol{\lambda}_h)_{h=1}^m, \mathbf{K}, \mathbf{L}\right) \\ \equiv &\left\{ \begin{array}{c} \left(\tilde{\mathbf{K}}, \tilde{\mathbf{L}}\right) \in Y \times Z : \\ \sum_{t=0}^T \left[p_t F\left(\tilde{K}_t, \tilde{L}_t\right) - r_t \tilde{K}_t - w_t \tilde{L}_t\right] \\ > \sum_{t=0}^T [p_t F(K_t, L_t) - r_t K_t - w_t L_t] \end{array} \right\} \end{aligned} \tag{14}$$

We observe that $\varphi_i : \Phi \to 2^{\Phi_i}$ where: $\Phi_0 \equiv P$, $\Phi_i \equiv X_i \times Y_i \times Z_i$ for $i = 1, \ldots, m$, $\Phi_{m+1} \equiv Y \times Z$ and $\Phi \equiv \Phi_0 \times \ldots \times \Phi_{m+1}$. $2^{\Phi_i}$ denotes the set of subsets of $\Phi_i$.

The proof of the following claim is also supplied in Becker et al. (2012).

**Claim 11** *$\varphi_i$ is a lower semi-continuous convex-valued correspondence for $i = 0, \ldots, m+1$.*

Let us now simplify the notation: $\mathbf{v}_0 \equiv (\mathbf{p}, \mathbf{r}, \mathbf{w})$, $\mathbf{v}_i \equiv (\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i)$ for $i = 1, \ldots, m$, $\mathbf{v}_{m+1} \equiv (\mathbf{K}, \mathbf{L})$ and

$$\mathbf{v} \equiv (\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m, \mathbf{v}_{m+1})$$

We are now able to prove the main result of the paper, that is the existence of a general equilibrium, through the following lemma.

**Lemma 3** *(fixed-point) There exists $\bar{\mathbf{v}} \in \Phi$ such that $\varphi_i(\bar{\mathbf{v}}) = \varnothing$ for $i = 0, \ldots, m+1$.*

**Proof.** Definition and properties of correspondence $\varphi$ satisfy the assumptions of the Gale and Mas-Colell (1975) fixed-point theorem: there exists $\mathbf{v} \in \Phi$ such that either $\varphi_i(\mathbf{v}) = \varnothing$ or $\mathbf{v}_i \in \varphi_i(\mathbf{v})$ for $i = 0, \ldots, m+1$ (see Becker et al. (2012) for a short proof). We observe the following.

(1) By definition of $\varphi_0$ (the inequality in (13) is strict): $(\mathbf{p}, \mathbf{r}, \mathbf{w}) \notin \varphi_0(\mathbf{v})$.

(2) $(\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i) \notin P_i(\mathbf{c}_i, \boldsymbol{\lambda}_i) \times Y_i$ implies that $(\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i) \notin \varphi_i(\mathbf{v})$ for $i = 1, \ldots, m$.

(3) By definition of $\varphi_{m+1}$ (the inequality in (14) is also strict): $(\mathbf{K}, \mathbf{L}) \notin \varphi_{m+1}(\mathbf{v})$.

Then, for $i = 0, \ldots, m+1$, $\mathbf{v}_i \notin \varphi_i(\mathbf{v})$. According to the Gale and Mas-Colell (1975) fixed-point theorem, there exists $\bar{\mathbf{v}} \in \Phi$ such that $\varphi_i(\bar{\mathbf{v}}) = \varnothing$ for $i = 0, \ldots, m+1$. ∎

More explicitly, there exists $\bar{\mathbf{v}} \in \Phi$ such that the following holds.

Focus on agent $i = 0$. For every $(\mathbf{p}, \mathbf{r}, \mathbf{w}) \in P$,

$$\begin{aligned} & \sum_{t=0}^{T} (p_t - \bar{p}_t) \left( \sum_{i=1}^{m} \left[ \bar{c}_{it} + \bar{k}_{it+1} - (1-\delta) \bar{k}_{it} \right] - F\left(\bar{K}_t, \bar{L}_t\right) \right) \\ & + \sum_{t=0}^{T} (r_t - \bar{r}_t) \left( \bar{K}_t - \sum_{i=1}^{m} \bar{k}_{it} \right) + \sum_{t=0}^{T} (w_t - \bar{w}_t) \left( \bar{L}_t - m + \sum_{i=1}^{m} \bar{\lambda}_{it} \right) \\ \leq \; & 0 \end{aligned} \tag{15}$$

Consider consumers $i = 1, \ldots, m$. $\left(\bar{\mathbf{c}}_i, \bar{\mathbf{k}}_i, \bar{\boldsymbol{\lambda}}_i\right) \in C_i(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}})$ and $B_i(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}}) \cap \left[P_i\left(\bar{\mathbf{c}}_i, \bar{\boldsymbol{\lambda}}_i\right) \times Y_i\right] = \varnothing$ for $i = 1, \ldots, m$. Then, for $i = 1, \ldots, m$, $(\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i) \in C_i(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}}) = \overline{B}_i(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}})$ implies

$$\sum_{t=0}^{T} \beta_i^t u_i(c_{it}, \lambda_{it}) \leq \sum_{t=0}^{T} \beta_i^t u_i\left(\bar{c}_{it}, \bar{\lambda}_{it}\right) \tag{16}$$

Focus on the firm $i = m+1$. For $t = 0, \ldots, T$ and for every $(\mathbf{K}, \mathbf{L}) \in Y \times Z$, we have $\sum_{t=0}^{T} \left[\bar{p}_t F(K_t, L_t) - \bar{r}_t K_t - \bar{w}_t L_t\right] \leq \sum_{t=0}^{T} \left[\bar{p}_t F\left(\bar{K}_t, \bar{L}_t\right) - \bar{r}_t \bar{K}_t - \bar{w}_t \bar{L}_t\right]$.

This is possible if and only if

$$\bar{p}_t F(K_t, L_t) - \bar{r}_t K_t - \bar{w}_t L_t \leq \bar{p}_t F(\bar{K}_t, \bar{L}_t) - \bar{r}_t \bar{K}_t - \bar{w}_t \bar{L}_t \quad (17)$$

for any $t$ (simply choose $(\mathbf{K}, \mathbf{L})$ such that $(K_s, L_s) = (\bar{K}_s, \bar{L}_s)$ if $s \neq t$, to prove the necessity, and sum (17) side by side to prove the sufficiency). In particular, the equilibrium profit is nonnegative.

$$\bar{p}_t F(\bar{K}_t, \bar{L}_t) - \bar{r}_t \bar{K}_t - \bar{w}_t \bar{L}_t \geq 0 \quad (18)$$

The proofs of Claims 12 to 14 in the following can be found in Becker et al. (2012). Claim 12 shows that input excess demands are nonnegative too.

**Claim 12** *If $\bar{p}_t > 0$, then $\bar{K}_t - \sum_{i=1}^m \bar{k}_{it} \geq 0$ and $\bar{L}_t - \sum_{i=1}^m \bar{l}_{it} \geq 0$.*

Let $\bar{Z}_t \equiv \sum_{i=1}^m \left[\bar{c}_{it} + \bar{k}_{it+1} - (1-\delta)\bar{k}_{it}\right] - F(\bar{K}_t, \bar{L}_t)$ be the aggregate excess demand for goods at time $t$. We want to prove that $\bar{Z}_t = 0$. Assume, by contradiction, that

$$\bar{Z}_t \neq 0 \quad (19)$$

**Claim 13** *If $\bar{Z}_t \neq 0$ and $p_t \bar{Z}_t \leq \bar{p}_t \bar{Z}_t$ for every $p_t$ with $|p_t| \leq 1$, then (1) $|\bar{p}_t| = 1$ and (2) $\bar{p}_t \bar{Z}_t > 0$.*

**Claim 14** *If $\bar{Z}_t \neq 0$, then $\bar{Z}_t > 0$ and, hence, $\bar{p}_t = 1$.*

Let us show eventually that good and input markets clear through Propositions 4 and 6.

**Proposition 4** *The goods market clears: $\bar{Z}_t = 0$, that is*

$$\sum_{i=1}^m \left[\bar{c}_{it} + \bar{k}_{it+1} - (1-\delta)\bar{k}_{it}\right] = F(\bar{K}_t, \bar{L}_t)$$

**Proof.** Assume, by contradiction, $\bar{Z}_t \neq 0$. Claim 14 implies $\bar{p}_t = 1$ and $\bar{Z}_t > 0$. It is possible to show that some budget constraint is violated (see Becker et al. (2012) for details). ■

We observe that the aggregate consumption lies within the bound: $\sum_{i=1}^m \bar{c}_{it} < B_c$ (see Becker et al. (2012) for details).

The end of the proof of equilibrium existence rests on Propositions 5 and 8 (prices positivity), 6 (market clearing for capital and labor), 7 (zero-profit condition), 9 (budget constraint). All the proofs are provided in Becker et al. (2012).

The good price is positive over time.

**Proposition 5** *$\bar{p}_t > 0$, $t = 0, \ldots, T$.*

Market clearing for capital and the labor rests on this price positivity.

**Proposition 6** *$\bar{K}_t = \sum_{i=1}^m \bar{k}_{it}$ and $\bar{L}_t = \sum_{i=1}^m \bar{l}_{it}$.*

To prove the zero-profit condition and input prices positivity, we observe that $\sum_{i=1}^{m} \bar{k}_{it} \leq A < B_K$ and $\sum_{i=1}^{m} \bar{l}_{it} \leq m < B_L$.

**Proposition 7** *At the prices* $(\bar{p}_t, \bar{r}_t, \bar{w}_t)$*, the equilibrium demand* $(\bar{K}_t, \bar{L}_t)$ *satisfies the zero-profit condition:* $\bar{p}_t F(\bar{K}_t, \bar{L}_t) = \bar{r}_t \bar{K}_t + \bar{w}_t \bar{L}_t$.

The other prices (the interest rate and the wage bill) turn out to be positive too.

**Proposition 8** $\bar{r}_t > 0,\ \bar{w}_t > 0,\ t = 0, \ldots, T$.

We conclude Appendix 1, noticing that, at equilibrium, the artificial budget constraint à la Bergstrom (1976) we needed to apply the Gale and Mas-Colell (1975) fixed-point argument, actually collapses in the ordinary one.

**Proposition 9** *The modified budget constraint at equilibrium is a budget constraint:* $\gamma(\bar{p}_t, \bar{r}_t, \bar{w}_t) = 0$ *for* $t = 0, \ldots, T$.

Proposition 9 concludes the proof of Proposition 1: for the finite-horizon bounded economy $\mathcal{E}^T, \bar{\mathbf{v}}$ is actually an equilibrium.

# 10 Appendix 2: existence of equilibrium in an infinite-horizon economy

We want to prove Theorem 5. From now on, any variable $x_t^T$ with subscript $t$ and superscript $T$ will refer to period $t$ in a $T$-truncated economy with $x_t^T = 0$ for $t > T$. As above, sequences will be denoted in bold type.

Under the Assumptions 1, 2, 3 and 5, an equilibrium

$$\left(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}}, \left(\bar{\mathbf{c}}_i, \bar{\mathbf{k}}_i, \bar{\boldsymbol{\lambda}}_i\right)_{i=1}^{m}, \bar{\mathbf{K}}, \bar{\mathbf{L}}\right)^T$$

of a truncated economy exists. Under these assumptions, namely separability and differentiability of preferences, the Kuhn-Tucker necessary conditions hold for the existence of an equilibrium in a truncated economy.

**Claim 15** *Under Assumption 5, the equilibrium of a truncated economy satisfies the following conditions.*

*For* $t = 0, \ldots, T$:

*(1)* $\bar{p}_t^T, \bar{r}_t^T, \bar{w}_t^T > 0$ *with* $\bar{p}_t^T + \bar{r}_t^T + \bar{w}_t^T = 1$ *(normalization),*

*(2)* $(\partial F / \partial K_t)\left(\bar{K}_t^T, \bar{L}_t^T\right) = \bar{r}_t^T / \bar{p}_t^T$,

*(3)* $(\partial F / \partial L_t)\left(\bar{K}_t^T, \bar{L}_t^T\right) = \bar{w}_t^T / \bar{p}_t^T$,

*(4)* $\bar{K}_t^T = \sum_{i=1}^{m} \bar{k}_{it}^T$,

*(5)* $\bar{L}_t^T = \sum_{i=1}^{m} \bar{l}_{it}^T$,

*(6)* $\sum_{i=1}^{m} \left[\bar{c}_{it}^T + \bar{k}_{it+1}^T - (1-\delta)\bar{k}_{it}^T\right] = F\left(\bar{K}_t^T, \bar{L}_t^T\right)$ *with* $\bar{k}_{iT+1}^T = 0$.

*For* $i = 1, \ldots, m,\ t = 0, \ldots, T$:

*(7)* $\beta_i^t u_i'\left(\bar{c}_{it}^T\right) = \bar{\mu}_{it}^T \bar{p}_t^T \geq \bar{\mu}_{it+1}^T \bar{p}_{t+1}^T (1-\delta) + \bar{\mu}_{it+1}^T \bar{r}_{t+1}^T$*, with equality when* $\bar{k}_{it+1}^T > 0$,

*(8)* $v_i'\left(\bar{\lambda}_{it}^T\right) \geq u_i'\left(\bar{c}_{it}^T\right)\bar{w}_t^T/\bar{p}_t^T$*, with equality when* $\bar{\lambda}_{it}^T < 1$*,*

*(9)* $\bar{p}_t^T\left[\bar{c}_{it}^T + \bar{k}_{it+1}^T - (1-\delta)\bar{k}_{it}^T\right] = \bar{r}_t^T\bar{k}_{it}^T + \bar{w}_t^T\left(1-\bar{\lambda}_{it}^T\right)$ *with* $\bar{k}_{it}^T \geq 0$*,* $\bar{k}_{iT+1}^T = 0$ *and* $0 \leq \bar{\lambda}_{it}^T \leq 1$*,*

*where* $\bar{\mu}_{it}^T$ *is the multiplier associated to the budget constraint at time* $t$*.*

**Proof.** See Bosi and Seegmuller (2010). ■

In the rest of Appendix 2, from Claim 16 to 26, we will prove that the equilibrium variables (prices and quantities) of the sequence of truncated economies converge to limit values that are actually equilibrium values of the limit economy. For simplicity, we will omit any reference to Assumptions 1, 2, 3 and 5. We will suppose that they are always satisfied.

Let us simplify the notation:

$$\begin{array}{llll} \bar{\zeta}_{it}^T \equiv \beta_i^t u_i'\left(\bar{c}_{it}^T\right)\bar{c}_{it}^T & \text{if } t \leq T, & \text{and} & \bar{\zeta}_{it}^T = 0 \quad \text{if } t > T, \\ \bar{\eta}_{it}^T \equiv \beta_i^t v_i'\left(\bar{\lambda}_{it}^T\right)\bar{\lambda}_{it}^T & \text{if } t \leq T, & \text{and} & \bar{\eta}_{it}^T = 0 \quad \text{if } t > T, \\ \bar{\theta}_{it}^T \equiv \beta_i^t v_i'\left(\bar{\lambda}_{it}^T\right) & \text{if } t \leq T, & \text{and} & \bar{\theta}_{it}^T = 0 \quad \text{if } t > T, \\ \bar{\vartheta}_{it}^T \equiv \bar{\mu}_{it}^T\bar{w}_t^T & \text{if } t \leq T, & \text{and} & \bar{\vartheta}_{it}^T = 0 \quad \text{if } t > T, \end{array} \tag{20}$$

and $\bar{\varepsilon}_{it}^T \equiv \bar{\theta}_{it}^T - \bar{\vartheta}_{it}^T$.

We notice that points (7) and (8) of Claim 15 entail $\bar{\varepsilon}_{it}^T \geq 0$ with $\bar{\varepsilon}_{it}^T = 0$ when $\bar{\lambda}_{it}^T < 1$.

The proofs of Claims 16 to 20 in the following are provided in Becker et al. (2012).

**Claim 16** *For any* $\varepsilon > 0$*, there exists* $\tau$ *such that, for any* $s > \tau$ *and any* $T$*,* $\sum_{t=s}^{\infty}\bar{\zeta}_{it}^T < \varepsilon$*.*

**Claim 17** *For any* $\varepsilon > 0$*, there exists* $\tau$ *such that, for any* $s > \tau$ *and any* $T$*,* $\sum_{t=s}^{\infty}\bar{\eta}_{it}^T < \varepsilon$*.*

**Claim 18** *For any* $\varepsilon > 0$*, there exists* $\tau$ *such that, for any* $s > \tau$ *and any* $T$*,* $\sum_{t=s}^{\infty}\bar{\vartheta}_{it}^T\bar{\lambda}_{it}^T < \varepsilon$ *and* $\sum_{t=s}^{\infty}\bar{\varepsilon}_{it}^T < \varepsilon$*. In addition, for any* $T$*,* $\left(\bar{\vartheta}_{it}^T\bar{\lambda}_{it}^T\right)_{t=0}^{\infty} \in l_+^1$ *and* $\left(\bar{\varepsilon}_{it}^T\right)_{t=0}^{\infty} \in l_+^1$*.*

**Claim 19** *For any* $\varepsilon > 0$*, there exists* $\tau$ *such that for any* $s > \tau$ *and any* $T \geq s$ *we have* $\sum_{t=s}^{T}\bar{\vartheta}_{it}^T < \varepsilon$*. In addition, for any* $T$*,*

$$\sum_{t=0}^{T}\bar{\vartheta}_{it}^T < \frac{u_i(A) + v_i(1)}{1-\beta_i} \tag{21}$$

**Claim 20** *Let* $\bar{\boldsymbol{\vartheta}}_i^T \equiv \left(\bar{\vartheta}_{it}^T\right)_{t=0}^{\infty}$*. There is a subsequence* $\left(\bar{\boldsymbol{\vartheta}}_i^{T_S}\right)_{S=0}^{\infty}$ *which converges for the* $l^1$*-topology to a sequence* $\bar{\boldsymbol{\vartheta}}_i \equiv \left(\bar{\vartheta}_{it}\right)_{t=0}^{\infty} \in l_+^1$*. The limit* $\bar{\boldsymbol{\vartheta}}_i$ *shares* *the same properties of the terms $\bar{\boldsymbol{\vartheta}}_i^T$ of the sequence, namely, (1) for any $\varepsilon > 0$ there exists $\tau$ (the same for all the terms) such that, for any $s > \tau$, we have $\sum_{t=s}^{\infty} \bar{\vartheta}_{it} \leq \varepsilon$, and (2) $\sum_{t=0}^{\infty} \bar{\vartheta}_{it} \leq [u_i(A) + v_i(1)] / (1 - \beta_i)$.*

We observe that the critical $\tau$ in Claims 16 to 20 are independent of $T$.

**Claim 21** *In the infinite-horizon economy, leisure demand is positive:*

$$\lim_{T \to \infty} \bar{\lambda}_{it}^T = \bar{\lambda}_{it} \in (0, 1]$$

**Proof.** We have $\bar{\theta}_{it}^T = \bar{\vartheta}_{it}^T + \bar{\varepsilon}_{it}^T$ with $\bar{\varepsilon}_{it}^T \geq 0$ and $\bar{\varepsilon}_{it}^T = 0$ if $\bar{\lambda}_{it}^T < 1$.

From Claim 19, we know that, for any $\varepsilon > 0$, there exists $\tau_1$ such that, for any $s > \tau_1$ and any $T$, $\sum_{t=s}^{\infty} \bar{\vartheta}_{it}^T \leq \varepsilon/2$.

From Claim 18, we know that for any $\varepsilon > 0$, there exists $\tau_2$ such that, for any $s > \tau_2$ and any $T$, $\sum_{t=s}^{\infty} \bar{\varepsilon}_{it}^T < \varepsilon/2$.

Hence, for any $\varepsilon > 0$, there exists $\tau \equiv \max\{\tau_1, \tau_2\}$ such that, for any $s > \tau$ and any $T$, $\sum_{t=s}^{\infty} \bar{\theta}_{it}^T = \sum_{t=s}^{\infty} \bar{\vartheta}_{it}^T + \sum_{t=s}^{\infty} \bar{\varepsilon}_{it}^T < \varepsilon$. In addition, for any $T$,

$$\sum_{t=0}^{\infty} \bar{\theta}_{it}^T = \sum_{t=0}^{\infty} \bar{\vartheta}_{it}^T + \sum_{t=0}^{\infty} \bar{\varepsilon}_{it}^T \leq \frac{u_i(A) + v_i(1)}{1 - \beta_i} + \frac{v_i(1)}{1 - \beta_i}$$

Let $\bar{\boldsymbol{\theta}}_i^T \equiv \left(\bar{\theta}_{it}^T\right)$. Then $\bar{\boldsymbol{\theta}}_i^T \to \bar{\boldsymbol{\theta}}_i \in l_+^1$ for the $l^1$-topology.

Therefore, for any $t$, $\bar{\theta}_{it}^T$ converges to $\bar{\theta}_{it} \in (0, +\infty)$. Hence, $\bar{\lambda}_{it}^T$ converges to $\bar{\lambda}_{it} > 0$ since $v_i$ satisfies the Inada conditions (Assumption 5). Clearly, $\bar{\lambda}_{it} \leq 1$. ■

**Claim 22** *In the infinite-horizon economy, the equilibrium prices are positive: $\lim_{T\to\infty} \bar{p}_t^T = \bar{p}_t \in (0,1)$, $\lim_{T\to\infty} \bar{r}_t^T = \bar{r}_t \in (0,1)$, $\lim_{T\to\infty} \bar{w}_t^T = \bar{w}_t \in (0,1)$.*

**Proof.** Focus on prices.

Suppose that $\lim_{T\to\infty} \bar{p}_t^T = 0$. We know that $\beta_i^t u_i'\left(\bar{c}_{it}^T\right) = \bar{\mu}_{it}^T \bar{p}_t^T$.

If $\bar{\mu}_{it}^T$ is bounded, we have $\lim_{T\to\infty} u_i'\left(\bar{c}_{it}^T\right) = 0$ which is impossible because $\bar{c}_{it}^T \leq A$ for every $T$.

Then, $\lim_{T\to\infty} \bar{\mu}_{it}^T = +\infty$. However, $\beta_i^t v_i'\left(\bar{\lambda}_{it}^T\right) / \bar{\mu}_{it}^T = \bar{w}_t^T + \bar{\varepsilon}_{it}^T / \bar{\mu}_{it}^T$ and $\lim_{T\to\infty} \bar{w}_t^T = \lim_{T\to\infty} \left(\bar{\theta}_{it}^T / \bar{\mu}_{it}^T\right) - \lim_{T\to\infty} \left(\bar{\varepsilon}_{it}^T / \bar{\mu}_{it}^T\right) = 0$ (Claim 21).

Since $\lim_{T\to\infty} \bar{p}_t^T = 0$, $\lim_{T\to\infty} \bar{w}_t^T = 0$ and $\bar{p}_t^T + \bar{w}_t^T + \bar{r}_t^T = 1$, we get $\lim_{T\to\infty} \bar{r}_t^T = 1$.

We know that $\bar{\mu}_{it-1}^T \bar{p}_{t-1}^T \geq \bar{\mu}_{it}^T \bar{p}_t^T (1 - \delta) + \bar{\mu}_{it}^T \bar{r}_t^T \geq \bar{\mu}_{it}^T \bar{r}_t^T$ (point (7) of Claim 15). Then $\lim_{T\to\infty} \bar{\mu}_{it-1}^T \bar{p}_{t-1}^T \geq \lim_{T\to\infty} \bar{\mu}_{it}^T \bar{r}_t^T = +\infty$.

Similarly, $\bar{\mu}_{it-2}^T \bar{p}_{t-2}^T \geq \bar{\mu}_{it-1}^T \bar{p}_{t-1}^T (1 - \delta) + \bar{\mu}_{it-1}^T \bar{r}_{t-1}^T \geq \bar{\mu}_{it-1}^T \bar{p}_{t-1}^T (1 - \delta)$ and $\lim_{T\to\infty} \bar{\mu}_{it-2}^T \bar{p}_{t-2}^T \geq \lim_{T\to\infty} \bar{\mu}_{it-1}^T \bar{p}_{t-1}^T (1 - \delta) = +\infty$.

Computing backward, we obtain $\lim_{T\to\infty} \bar{\mu}_{i0}^T \bar{p}_0^T = +\infty$.

If $\lim_{T\to\infty} \bar{p}_0^T > 0$, since $\bar{p}_0^T \leq 1$, then $\lim_{T\to\infty} \bar{\mu}_{i0}^T = +\infty$ and, since $\lim_{T\to\infty} \bar{\mu}_{i0}^T \bar{w}_0^T = \bar{\vartheta}_{i0} < +\infty$, this implies $\lim_{T\to\infty} \bar{w}_0^T = 0$. Thus,

$$0 = \bar{p}_0 F\left(K_0, \bar{L}_0\right) - \bar{r}_0 K_0 - \bar{w}_0 \bar{L}_0 = \bar{p}_0 F\left(K_0, \bar{L}_0\right) - \bar{r}_0 K_0$$

Choose $L_0 > \bar{L}_0$ in order to obtain a higher profit in contradiction with profit maximization.

Let $\lim_{T\to\infty} \bar{p}_0^T = 0$. We know that $u_i'(A) \leq u_i'\left(\bar{c}_{i0}^T\right) = \beta_i^0 u_i'\left(\bar{c}_{i0}^T\right) = \bar{\mu}_{i0}^T \bar{p}_0^T$.

If $\lim_{T\to\infty} \bar{\mu}_{i0}^T < +\infty$, we have $\lim_{T\to\infty} \bar{\mu}_{i0}^T \bar{p}_0^T = 0$ and $u_i'(A) \leq 0$, a contradiction.

If $\lim_{T\to\infty} \bar{\mu}_{i0}^T = +\infty$, then $\lim_{T\to\infty} \bar{\mu}_{i0}^T \bar{w}_0^T = \bar{\vartheta}_{i0} < +\infty$ gives $\lim_{T\to\infty} \bar{w}_0^T = 0$ and $\lim_{T\to\infty} \bar{r}_0^T = 1$. Focus on the first budget constraint:

$$\bar{p}_0^T \left[\bar{c}_{i0}^T + \bar{k}_{i1}^T - (1-\delta) k_{i0}\right] = \bar{r}_0^T k_{i0} + \bar{w}_0^T \left(1 - \bar{\lambda}_{i0}^T\right)$$

Assumption 3 ensures $k_{i0} > 0$. In this case, in the limit:

$$0 = \bar{p}_0 \left[\bar{c}_{i0} + \bar{k}_{i1} - (1-\delta) k_{i0}\right] = \bar{r}_0 k_{i0} + \bar{w}_0 \left(1 - \bar{\lambda}_{i0}\right) \geq k_{i0} > 0$$

a contradiction. Thus, for every $t$, $\bar{p}_t^T \to \bar{p}_t > 0$.

Focus now on $\bar{r}_t$ and $\bar{w}_t$. In the limit, $\bar{p}_t F\left(\bar{K}_t, \bar{L}_t\right) - \bar{r}_t \bar{K}_t - \bar{w}_t \bar{L}_t = 0$.

If $\bar{r}_t = 0$ (respectively, $\bar{w}_t = 0$), then $\bar{p}_t F\left(\bar{K}_t, \bar{L}_t\right) - \bar{w}_t \bar{L}_t = 0$ (respectively, $\bar{p}_t F\left(\bar{K}_t, \bar{L}_t\right) - \bar{r}_t \bar{K}_t = 0$). Fix $L_t > 0$ ($K_t > 0$) and choose $K_t$ ($L_t$) large enough such that $\bar{p}_t F(K_t, L_t) - \bar{w}_t L_t > 0$ ($\bar{p}_t F(K_t, L_t) - \bar{r}_t K_t > 0$), against the equilibrium condition.

Thus, $\bar{p}_t, \bar{r}_t, \bar{w}_t > 0$. ■

**Claim 23** $\bar{c}_{it} = \lim_{T\to\infty} \bar{c}_{it}^T \in (0, +\infty)$.

**Proof.** For any $t$, $\sum_{i=1}^m \bar{c}_{it}^T \leq A$. This implies $\bar{c}_{it}^T \leq A$ independently of the choice of $T$ and $\lim_{T\to\infty} \bar{c}_{it}^T \leq A < +\infty$. In addition, if $\bar{c}_{it} = \lim_{T\to\infty} \bar{c}_{it}^T = 0$, then, since $u_i'\left(\bar{c}_{it}^T\right) \bar{w}_t^T / \bar{p}_t^T \leq v_i'\left(\bar{\lambda}_{it}^T\right)$, we obtain $+\infty = \lim_{T\to\infty} u_i'\left(\bar{c}_{it}^T\right) \bar{w}_t^T / \bar{p}_t^T \leq \lim_{T\to\infty} v_i'\left(\bar{\lambda}_{it}^T\right)$, that is $\bar{\lambda}_{it} = \lim_{T\to\infty} \bar{\lambda}_{it}^T = 0$, a contradiction (see Claim 21). Then, $\bar{c}_{it} > 0$. ■

**Claim 24** *For any $t$, $\lim_{T\to\infty} \bar{K}_t^T = \bar{K}_t > 0$ and $\lim_{T\to\infty} \bar{L}_t^T = \bar{L}_t > 0$.*

**Proof.** We know that $\sum_{i=1}^m \bar{k}_{it+1} \geq 0$ and that $\sum_{i=1}^m \bar{c}_{it} + \sum_{i=1}^m \bar{k}_{it+1} = F\left(\bar{K}_t, \bar{L}_t\right) + (1-\delta) \bar{K}_t$. If $\bar{K}_t = 0$, then $\bar{c}_{it} = 0$ for every $i$, a contradiction. Now, if $\bar{L}_t = 0$, we have $\bar{r}_t \bar{K}_t = 0$ and, hence, $\bar{K}_t = 0$: a contradiction. ■

The transversality condition holds.

**Claim 25** $\lim_{t\to+\infty} \bar{\mu}_{it} \bar{p}_t \bar{k}_{it+1} = 0$.

**Proof.** Let $\varepsilon > 0$. We know that there exists $\tau$ such that for any pair $(s, s')$ such that $s' > s > \tau$, and any $T > s$, we have $\sum_{t=s}^{s'} \bar{\zeta}_{it}^T < \varepsilon$ and $\sum_{t=s}^{s'} \bar{\vartheta}_{it}^T \left(1 - \bar{\lambda}_{it}^T\right) < \varepsilon$ for every $i$ (see Claim 19). Taking the limit for $T \to +\infty$, we find

$$\begin{aligned} \varepsilon &\geq \lim_{T\to+\infty} \sum_{t=s}^{s'} \bar{\zeta}_{it}^T = \sum_{t=s}^{s'} \lim_{T\to+\infty} \left[\beta_i^t u_i' \left(\bar{c}_{it}^T\right) \bar{c}_{it}^T\right] = \sum_{t=s}^{s'} \beta_i^t u_i' \left(\bar{c}_{it}\right) \bar{c}_{it} \\ &= \sum_{t=s}^{s'} \bar{\mu}_{it} \bar{p}_t \bar{c}_{it} \end{aligned}$$

(see Claim 23) and

$$\begin{aligned} \varepsilon &\geq \lim_{T\to+\infty} \sum_{t=s}^{s'} \bar{\vartheta}_{it}^T \left(1 - \bar{\lambda}_{it}^T\right) = \sum_{t=s}^{s'} \lim_{T\to+\infty} \left(\bar{\mu}_{it}^T \bar{w}_t^T\right) \left(1 - \lim_{T\to+\infty} \bar{\lambda}_{it}^T\right) \\ &= \sum_{t=s}^{s'} \bar{\mu}_{it} \bar{w}_t \left(1 - \bar{\lambda}_{it}\right) \end{aligned}$$

(see Claims 20 and 21). Since this holds for any $s' > s$, we get also

$$\sum_{t=s}^{\infty} \bar{\mu}_{it} \bar{p}_t \bar{c}_{it} \leq \varepsilon \text{ and } \sum_{t=s}^{\infty} \bar{\mu}_{it} \bar{w}_t \left(1 - \bar{\lambda}_{it}\right) \leq \varepsilon \tag{22}$$

From the budget constraints, for any $s' \geq T$, we obtain

$$\begin{aligned} \varepsilon &> \sum_{t=s}^{s'} \bar{\mu}_{it}^T \bar{p}_t^T \bar{c}_{it}^T = \bar{\mu}_{is}^T \bar{p}_s^T (1-\delta) \bar{k}_{is}^T + \bar{\mu}_{is}^T \bar{r}_s^T \bar{k}_{is}^T + \sum_{t=s}^{s'} \bar{\vartheta}_{it}^T \left(1 - \bar{\lambda}_{it}^T\right) \\ &\geq \bar{\mu}_{is}^T \bar{p}_s^T (1-\delta) \bar{k}_{is}^T + \bar{\mu}_{is}^T \bar{r}_s^T \bar{k}_{is}^T \end{aligned}$$

Taking the limit for $T \to +\infty$, we find $\bar{\mu}_{is} \bar{p}_s (1-\delta) \bar{k}_{is} \leq \varepsilon$ and $\bar{\mu}_{is} \bar{r}_s \bar{k}_{is} \leq \varepsilon$ for every $s > \tau$. This implies $\limsup_s \bar{\mu}_{is} \bar{p}_s (1-\delta) \bar{k}_{is} \leq \varepsilon$ and $\limsup_s \bar{\mu}_{is} \bar{r}_s \bar{k}_{is} \leq \varepsilon$. These inequalities hold for any $\varepsilon > 0$. Hence

$$\lim_{t\to+\infty} \bar{\mu}_{it} \bar{p}_t (1-\delta) \bar{k}_{it} = 0 \text{ and } \lim_{t\to+\infty} \bar{\mu}_{it} \bar{r}_t \bar{k}_{it} = 0 \tag{23}$$

Again, from the budget constraint, we have $\bar{\mu}_{it}^T \bar{p}_t^T \bar{k}_{it+1}^T = \bar{\mu}_{it}^T \bar{p}_t^T (1-\delta) \bar{k}_{it}^T + \bar{\mu}_{it}^T \bar{r}_t^T \bar{k}_{it}^T + \bar{\mu}_{it}^T \bar{w}_t^T \left(1 - \bar{\lambda}_{it}^T\right) - \bar{\mu}_{it}^T \bar{p}_t^T \bar{c}_{it}^T$. Taking the limit for $T \to +\infty$, we obtain $\bar{\mu}_{it} \bar{p}_t \bar{k}_{it+1} = \bar{\mu}_{it} \bar{p}_t (1-\delta) \bar{k}_{it} + \bar{\mu}_{it} \bar{r}_t \bar{k}_{it} + \bar{\mu}_{it} \bar{w}_t \left(1 - \bar{\lambda}_{it}\right) - \bar{\mu}_{it} \bar{p}_t \bar{c}_{it}$. We know that $\lim_{t\to+\infty} \bar{\mu}_{it} \bar{p}_t (1-\delta) \bar{k}_{it} = 0$ and $\lim_{t\to+\infty} \bar{\mu}_{it} \bar{r}_t \bar{k}_{it} = 0$ (see (23)). We know also that $\lim_{t\to+\infty} \bar{\mu}_{it} \bar{w}_t \left(1 - \bar{\lambda}_{it}\right) = 0$ and $\lim_{t\to+\infty} \bar{\mu}_{it} \bar{p}_t \bar{c}_{it} = 0$ (see (22)). Therefore, $\lim_{t\to+\infty} \bar{\mu}_{it} \bar{p}_t \bar{k}_{it+1} = 0$. ■

Let us eventually prove that an equilibrium exists in the infinite-horizon economy.

**Claim 26** $\left(\bar{\mathbf{p}}, \bar{\mathbf{r}}, \bar{\mathbf{w}}, \left(\bar{\mathbf{c}}_i, \bar{\mathbf{k}}_i, \bar{\boldsymbol{\lambda}}_i\right)_{i=1}^m, \bar{\mathbf{K}}, \bar{\mathbf{L}}\right)$ *is an equilibrium.*

**Proof.** Consider first the firm. For every truncated $T$-economy, a zero profit condition holds: $\bar{p}_t^T F\left(\bar{K}_t^T, \bar{L}_t^T\right) - \bar{r}_t^T \bar{K}_t^T - \bar{w}_t^T \bar{L}_t^T = 0$. In the limit, for the infinite-horizon economy: $\bar{p}_t F\left(\bar{K}_t, \bar{L}_t\right) - \bar{r}_t \bar{K}_t - \bar{w}_t \bar{L}_t = 0$, because $\bar{p}_t^T \to \bar{p}_t \in (0,1)$, $\bar{r}_t^T \to \bar{r}_t \in (0,1)$, $\bar{w}_t \to \bar{w}_t \in (0,1)$, $\bar{K}_t^T = \sum_{i=1}^m \bar{k}_{it}^T \to \sum_{i=1}^m \bar{k}_{it} = \bar{K}_t < +\infty$, $\bar{L}_t^T = \sum_{i=1}^m \bar{l}_{it}^T \to \sum_{i=1}^m \bar{l}_{it} = \bar{L}_t < +\infty$. If $\left(\bar{K}_t, \bar{L}_t\right)$ does not maximize the profit in the infinite-horizon economy, then there exists $(K_t, L_t)$ such that $\bar{p}_t F\left(K_t, L_t\right) - \bar{r}_t K_t - \bar{w}_t L_t > \bar{p}_t F\left(\bar{K}_t, \bar{L}_t\right) - \bar{r}_t \bar{K}_t - \bar{w}_t \bar{L}_t = 0$ and, so, a critical $\tau$, such that, for any $T > \tau$, $\bar{p}_t^T F\left(K_t, L_t\right) - \bar{r}_t^T K_t - \bar{w}_t^T L_t > \bar{p}_t^T F\left(\bar{K}_t^T, \bar{L}_t^T\right) - \bar{r}_t^T \bar{K}_t^T - \bar{w}_t^T \bar{L}_t^T = 0$ against the fact that $\left(\bar{K}_t^T, \bar{L}_t^T\right)$ maximizes the profit in the $T$-economy.

Focus now on consumers. Consider an alternative sequence $(\mathbf{c}_i, \mathbf{k}_i, \boldsymbol{\lambda}_i)$ with $k_{i0} = \bar{k}_{i0}$, which satisfies the budget constraints in the infinite-horizon economy. We have

$$
\begin{aligned}
\Delta_T &\equiv \sum_{t=0}^T \beta_i^t \left[u_i\left(\bar{c}_{it}\right) + v_i\left(\bar{\lambda}_{it}\right)\right] - \sum_{t=0}^T \beta_i^t \left[u_i\left(c_{it}\right) + v_i\left(\lambda_{it}\right)\right] \\
&= \sum_{t=0}^T \beta_i^t \left[u_i\left(\bar{c}_{it}\right) - u_i\left(c_{it}\right)\right] + \sum_{t=0}^T \beta_i^t \left[v_i\left(\bar{\lambda}_{it}\right) - v_i\left(\lambda_{it}\right)\right] \\
&\geq \sum_{t=0}^T \beta_i^t u_i'\left(\bar{c}_{it}\right)\left(\bar{c}_{it} - c_{it}\right) + \sum_{t=0}^T \beta_i^t v_i'\left(\bar{\lambda}_{it}\right)\left(\bar{\lambda}_{it} - \lambda_{it}\right) \\
&\geq \sum_{t=0}^T \bar{\mu}_{it} \bar{p}_t \left(\bar{c}_{it} - c_{it}\right) + \sum_{t=0}^T \bar{\mu}_{it} \bar{w}_t \left(\bar{\lambda}_{it} - \lambda_{it}\right)
\end{aligned}
$$

We observe that

$$
\begin{aligned}
\bar{\mu}_{it}\bar{p}_t\bar{c}_{it} - \bar{\mu}_{it}\bar{w}_t\left(1 - \bar{\lambda}_{it}\right) &= \bar{\mu}_{it}\bar{r}_t\bar{k}_{it} + \bar{\mu}_{it}\bar{p}_t\left(1-\delta\right)\bar{k}_{it} - \bar{\mu}_{it}\bar{p}_t\bar{k}_{it+1} \\
\bar{\mu}_{it}\bar{p}_t c_{it} - \bar{\mu}_{it}\bar{w}_t\left(1 - \lambda_{it}\right) &\leq \bar{\mu}_{it}\bar{r}_t k_{it} + \bar{\mu}_{it}\bar{p}_t\left(1-\delta\right) k_{it} - \bar{\mu}_{it}\bar{p}_t k_{it+1}
\end{aligned}
$$

where the first equality holds because of the Kuhn-Tucker method.

Subtracting member by member, we get

$$
\begin{aligned}
& \bar{\mu}_{it}\bar{p}_t\left(\bar{c}_{it} - c_{it}\right) + \bar{\mu}_{it}\bar{w}_t\left(\bar{\lambda}_{it} - \lambda_{it}\right) \\
\geq\ & \left[\bar{\mu}_{it}\bar{r}_t\bar{k}_{it} + \bar{\mu}_{it}\bar{p}_t\left(1-\delta\right)\bar{k}_{it} - \bar{\mu}_{it}\bar{p}_t\bar{k}_{it+1}\right] \\
& - \left[\bar{\mu}_{it}\bar{r}_t k_{it} + \bar{\mu}_{it}\bar{p}_t\left(1-\delta\right) k_{it} - \bar{\mu}_{it}\bar{p}_t k_{it+1}\right]
\end{aligned}
$$

Summing over $t$, we obtain

$$\begin{aligned}
\Delta_T &\geq \sum_{t=0}^{T} \bar{\mu}_{it}\bar{p}_t\left(\bar{c}_{it}-c_{it}\right)+\sum_{t=0}^{T}\bar{\mu}_{it}\bar{w}_t\left(\bar{\lambda}_{it}-\lambda_{it}\right) \\
&\geq \sum_{t=0}^{T}\left[\bar{\mu}_{it}\bar{p}_t\left(1-\delta\right)\bar{k}_{it}+\bar{\mu}_{it}\bar{r}_t\bar{k}_{it}-\bar{\mu}_{it}\bar{p}_t\bar{k}_{it+1}\right] \\
&\quad -\sum_{t=0}^{T}\left[\bar{\mu}_{it}\bar{p}_t\left(1-\delta\right)k_{it}+\bar{\mu}_{it}\bar{r}_t k_{it}-\bar{\mu}_{it}\bar{p}_t k_{it+1}\right] \\
&= \sum_{t=0}^{T-1}\left[-\bar{\mu}_{it}\bar{p}_t+\bar{\mu}_{it+1}\bar{p}_{t+1}\left(1-\delta\right)+\bar{\mu}_{it+1}\bar{r}_{t+1}\right]\left(\bar{k}_{it+1}-k_{it+1}\right) \\
&\quad -\bar{\mu}_{iT}\bar{p}_T\bar{k}_{iT+1}+\bar{\mu}_{iT}\bar{p}_T k_{iT+1} \qquad (24)
\end{aligned}$$

because $k_{i0}=\bar{k}_{i0}$.

We have from Claim 15, point (7), for $t=0,\ldots,T-1$,

$$\bar{\mu}_{it}\bar{p}_t \geq \bar{\mu}_{it+1}\bar{p}_{t+1}\left(1-\delta\right)+\bar{\mu}_{it+1}\bar{r}_{t+1}$$

with equality when $\bar{k}_{it+1}>0$.

Thus, if $\bar{k}_{it+1}>0$, $-\bar{\mu}_{it}\bar{p}_t+\bar{\mu}_{it+1}\bar{p}_{t+1}\left(1-\delta\right)+\bar{\mu}_{it+1}\bar{r}_{t+1}=0$ and

$$\left[-\bar{\mu}_{it}\bar{p}_t+\bar{\mu}_{it+1}\bar{p}_{t+1}\left(1-\delta\right)+\bar{\mu}_{it+1}\bar{r}_{t+1}\right]\left(\bar{k}_{it+1}-k_{it+1}\right)=0$$

If $\bar{k}_{it+1}=0$, $\bar{\mu}_{it}\bar{p}_t-\bar{\mu}_{it+1}\bar{p}_{t+1}\left(1-\delta\right)-\bar{\mu}_{it+1}\bar{r}_{t+1}\geq 0$ and

$$\begin{aligned}
&\left[-\bar{\mu}_{it}\bar{p}_t+\bar{\mu}_{it+1}\bar{p}_{t+1}\left(1-\delta\right)+\bar{\mu}_{it+1}\bar{r}_{t+1}\right]\left(\bar{k}_{it+1}-k_{it+1}\right) \\
=\; & k_{it+1}\left[\bar{\mu}_{it}\bar{p}_t-\bar{\mu}_{it+1}\bar{p}_{t+1}\left(1-\delta\right)-\bar{\mu}_{it+1}\bar{r}_{t+1}\right]\geq 0
\end{aligned}$$

Thus, inequality (24) implies:

$$\Delta_T \geq -\bar{\mu}_{iT}\bar{p}_T\bar{k}_{iT+1}+\bar{\mu}_{iT}\bar{p}_T k_{iT+1}\geq -\bar{\mu}_{iT}\bar{p}_T\bar{k}_{iT+1}$$

From Claim 25, we obtain $\lim_{T\to+\infty}\Delta_T\geq 0$ and, finally,

$$\sum_{t=0}^{\infty}\beta_i^t\left[u_i\left(\bar{c}_{it}\right)+v_i\left(\bar{\lambda}_{it}\right)\right]\geq\sum_{t=0}^{\infty}\beta_i^t\left[u_i\left(c_{it}\right)+v_i\left(\lambda_{it}\right)\right]$$

Hence, $\left(\bar{\mathbf{c}}_i,\bar{\boldsymbol{\lambda}}_i\right)$ maximizes the consumer's objective. ■

# 11 Appendix 3

Let us prove Claim 6

**Proof.** For any truncated economy, the vector of equilibrium multipliers $\left(\bar{\mu}_{it}^T\right)_{t=0}^T$ exists and satisfies $\bar{\mu}_{it}^T = \beta_i^t u_i'\left(\bar{c}_{it}^T\right)/\bar{p}_t^T$ (see point (7) in Claim 15). Since $\lim_{T\to+\infty} \bar{c}_{it}^T = \bar{c}_{it} \in (0,+\infty)$ and $\lim_{T\to+\infty} \bar{p}_t^T = \bar{p}_t \in (0,+\infty)$, we find also

$$\bar{\mu}_{it} = \lim_{T\to+\infty} \frac{\beta_i^t u_i'\left(\bar{c}_{it}^T\right)}{\bar{p}_t^T} = \frac{\beta_i^t u_i'\left(\bar{c}_{it}\right)}{\bar{p}_t} \in (0,+\infty)$$

In addition, we obtain from point (7) in Claim 15:

$$\begin{aligned}\bar{\mu}_{it}\bar{p}_t &= \lim_{T\to+\infty}\left(\bar{\mu}_{it}^T\bar{p}_t^T\right)\\ &\geq \lim_{T\to+\infty}\left(\bar{\mu}_{it+1}^T\left[\bar{p}_{t+1}^T(1-\delta)+\bar{r}_{t+1}^T\right]\right) = \bar{\mu}_{it+1}\left[\bar{p}_{t+1}(1-\delta)+\bar{r}_{t+1}\right]\end{aligned}$$

because $\lim_{T\to+\infty} \bar{\mu}_{it}^T = \bar{\mu}_{it} \in (0,+\infty)$.

If $\lim_{T\to+\infty} \bar{k}_{it+1}^T = \bar{k}_{it+1} > 0$, there is $S$ such that, for any $T > S$, $\bar{k}_{it+1}^T > 0$ and $\bar{\mu}_{it}^T\bar{p}_t^T = \bar{\mu}_{it+1}^T\left[\bar{p}_{t+1}^T(1-\delta)+\bar{r}_{t+1}^T\right]$. Thus, $\bar{\mu}_{it}\bar{p}_t = \bar{\mu}_{it+1}\left[\bar{p}_{t+1}(1-\delta)+\bar{r}_{t+1}\right]$ if $k_{it+1} > 0$.

Summing up, we find that the first-order conditions of point (7) in Claim 15 are satisfied in the limit economy:

$$\begin{aligned}\bar{\mu}_{it}\bar{p}_t &= \beta_i^t u_i'\left(\bar{c}_{it}\right)\\ \bar{\mu}_{it}\bar{p}_t &\geq \bar{\mu}_{it+1}\bar{p}_{t+1}\left(1-\delta+\bar{\rho}_{t+1}\right)\\ \bar{\mu}_{it}\bar{p}_t &= \bar{\mu}_{it+1}\bar{p}_{t+1}\left(1-\delta+\bar{\rho}_{t+1}\right) \text{ if } \bar{k}_{it+1} > 0\end{aligned}$$

■

Let us prove now Proposition 2.

**Proof.** Assume that such a sequence exists. Let $\omega_t \equiv w_t/p_t$. We have just to show that

$$\lim_{T\to+\infty}\left[\sum_{t=0}^T \bar{Q}_t\left(\bar{C}_t - C_t\right) + \sum_{t=0}^T \bar{Q}_t\bar{\omega}_t\left(\sum_{i=1}^m \bar{\lambda}_{it} - \sum_{i=1}^m \lambda_{it}\right)\right] \geq 0$$

a contradiction.

It is enough to prove that feasibility and first-order conditions imply

$$\left[\sum_{t=0}^T \bar{Q}_t\left(\bar{C}_t - C_t\right) + \sum_{t=0}^T \bar{Q}_t\bar{\omega}_t\left(\sum_{i=1}^m \bar{\lambda}_{it} - \sum_{i=1}^m \lambda_{it}\right)\right] \geq -\bar{Q}_T\bar{K}_{T+1} \qquad (25)$$

Since capitals are uniformly bounded above, the result follows from $\lim_{T\to\infty} \bar{Q}_T = 0$.

Let us prove inequality (25). Using (12), we find

$$
\begin{aligned}
\Delta_T &\equiv \sum_{t=0}^{T} \bar{Q}_t \left(\bar{C}_t - C_t\right) + \sum_{t=0}^{T} \bar{Q}_t \bar{\omega}_t \left( \sum_{i=1}^{m} \bar{\lambda}_{it} - \sum_{i=1}^{m} \lambda_{it} \right) \\
&= \sum_{t=0}^{T} \bar{Q}_t \left[ F\left(\bar{K}_t, \bar{L}_t\right) + (1-\delta)\bar{K}_t - \bar{K}_{t+1} \right] \\
&\quad - \sum_{t=0}^{T} \bar{Q}_t \left[ F\left(K_t, L_t\right) + (1-\delta) K_t - K_{t+1} \right] \\
&\quad + \sum_{t=0}^{T} \bar{Q}_t \bar{\omega}_t \left(m - \bar{L}_t\right) - \sum_{t=0}^{T} \bar{Q}_t \bar{\omega}_t \left(m - L_t\right) \\
&= \sum_{t=0}^{T} \bar{Q}_t \left[ F\left(\bar{K}_t, \bar{L}_t\right) - F\left(K_t, L_t\right) + (1-\delta)\left(\bar{K}_t - K_t\right) - \left(\bar{K}_{t+1} - K_{t+1}\right) \right] \\
&\quad - \sum_{t=0}^{T} \bar{Q}_t \bar{\omega}_t \left(\bar{L}_t - L_t\right)
\end{aligned}
$$

$$
\begin{aligned}
&\geq \sum_{t=0}^{T} \bar{Q}_t \left[ F_K\left(\bar{K}_t, \bar{L}_t\right)\left(\bar{K}_t - K_t\right) + F_L\left(\bar{K}_t, \bar{L}_t\right)\left(\bar{L}_t - L_t\right) \right] \\
&\quad + (1-\delta) \sum_{t=0}^{T} \bar{Q}_t \left(\bar{K}_t - K_t\right) - \sum_{t=0}^{T} \bar{Q}_t \left(\bar{K}_{t+1} - K_{t+1}\right) - \sum_{t=0}^{T} \bar{Q}_t \bar{\omega}_t \left(\bar{L}_t - L_t\right) \\
&= \sum_{t=0}^{T} \bar{Q}_t \left[ \bar{\rho}_t \left(\bar{K}_t - K_t\right) + \bar{\omega}_t \left(\bar{L}_t - L_t\right) \right] - \sum_{t=0}^{T} \bar{Q}_t \bar{\omega}_t \left(\bar{L}_t - L_t\right) \\
&\quad + (1-\delta) \sum_{t=0}^{T} \bar{Q}_t \left(\bar{K}_t - K_t\right) - \sum_{t=0}^{T} \bar{Q}_t \left(\bar{K}_{t+1} - K_{t+1}\right) \\
&= \sum_{t=0}^{T} \bar{Q}_t \left(1 - \delta + \bar{\rho}_t\right)\left(\bar{K}_t - K_t\right) - \sum_{t=0}^{T} \bar{Q}_t \left(\bar{K}_{t+1} - K_{t+1}\right)
\end{aligned}
$$

because of points (2) and (3) in Claim 15). Replacing (7), we get:

$$\begin{aligned}
\Delta_T &\geq \bar{Q}_0 (1-\delta+\bar{\rho}_0) \left(\bar{K}_0 - K_0\right) + \sum_{t=1}^{T} \bar{Q}_t (1-\delta+\bar{\rho}_t) \left(\bar{K}_t - K_t\right) \\
&\quad - \sum_{t=0}^{T} \bar{Q}_t \left(\bar{K}_{t+1} - K_{t+1}\right) \\
&= \sum_{t=0}^{T-1} \bar{Q}_{t+1} \left(1-\delta+\bar{\rho}_{t+1}\right) \left(\bar{K}_{t+1} - K_{t+1}\right) \\
&\quad - \sum_{t=0}^{T-1} \bar{Q}_t \left(\bar{K}_{t+1} - K_{t+1}\right) - \bar{Q}_T \left(\bar{K}_{T+1} - K_{T+1}\right) \\
&\geq \sum_{t=0}^{T-1} \left[\bar{Q}_{t+1} \left(1-\delta+\bar{\rho}_{t+1}\right) - \bar{Q}_t\right] \left(\bar{K}_{t+1} - K_{t+1}\right) - \bar{Q}_T \bar{K}_{T+1} \\
&= -\bar{Q}_T \bar{K}_{T+1}
\end{aligned}$$

because $\bar{K}_0 = K_0$. ■

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