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Toward a Unified Growth Theory

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Abstract

This paper developes a unified growth model capturing issues of endogenous economic growth, fertility, infant mortality, technological progress, education and environmental quality to interpret the evolution of history from Malthusian stagnation regime, through the demographic transition to modern growth. The production structure in this paper allows to explain the so-called Environmental Kurznet Curve theoretically. The model suggests that in the very early stage of development characterized by small population, low education and low technological level, the economy is trapped in a low stable steady state with very low growth rate of technological progress, zero-education and hence low suriving probability of infant which makes the population growth very small. In this stagnation the technological progress makes the pollution more polluted. In the period of demographic transition, corresponding to the industrial revolution time in Western Europe when there is a sharp increase in technological level, the infant mortality rate increases due to the pollution and hence the fertility also increases due to maximizing utility behavior of households. The infant mortality and fertility rates start decreasing when the level of technology exceeds a threshold (normalized by 1) in which technology has positive effect on environmental quality. The growth rate of technological progress plays an crucial role in demographic transition. Investment in education increases along with the increase in growth of technological progress because education get more return via supplying human capital in the future. The more investment in education for children, the less number of children the houselds raise because of the budget contraints of household. This trade-off along with the growth of technological progress make the fertility rate decreasing, then the economy enters a Modern Growth regime with reduced population and sustained growth rate of income.

Keyworks: Technological progress, human capital, fertility, infant mortality
JEL: J13, O11, O33

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1. Introduction

The unified modeling of the long transition process including three distinct regimes, from thousands of years of Mathusian stagnation with high fertility and mortality rate, and low economic growth, through the demographic transition to modern growth with low fertility and mortality but sustained growth, is one of the most significant research challenges facing economists interested in growth and development. Galor and Weil (2000) develop a unified growth model that captures the historical evolution of population, technology, and output. It encompasses the endogenous transition between these three distinct regimes that have characterized economic development. The authors focus on the two most important differences between these regimes from a macroeconomic viewpoint: (i) in the behavior of income per capita; and (ii) in the relationship between the level of income per capita and the growth rate of population. Galor and Moav (2002) develop a similar model to develop an evolution growth theory that captures the interplay between the evolution of mankind and economic growth since the emergence of the human species. These papers, however, just take into account the growth rate of population through the net fertility rate, not separating between the infant mortality and fertility rate which in fact evolute in complex trajectories needed to explain.

The most basic description of the relationship between population growth and income was proposed by Thomas R. Malthus (1798). There are two key components in the Malthusian model. The first one is the existence of some factor of production, such as land, which is fixed in supplying, implying decreasing returns to scale of all other factors of production. The second one is a positive effect of the standard of living on the growth rate of population. According to Malthus, the standard of living will be high when population size is small, and population will grow as a natural result of passion between the sexes. When population size is large, the standard of living will be low and population will be reduced by the either the "preventive check" (intentional reduction of fertility) or by the "positive check" (malnutrition, disease, and femine) (see more discussion in Galor and Weil 2000, pp. 807). Malthus (1798) stated that "*... Population, when unchecked, increase in a geometrical ratio. Subsistence increases only in an arithmetical ratio...*". The Malthusian model implies that, in the absence of changes in technology or in the availability of land, the size of the population will be self-equilibrating. Furthermore, increases in available resources will be, in the long run, offset by increases in the size of population. The predictions of the Malthusian model are consistent with the evolution of technology, population, and output per capita in the history. For thousand of years, the living standard was nearly constant and did not differ considerably across countries. Estimations from Angus Maddison (1982) show that the growth rate of GDP per capita in Europe between 500 and 1500 was zero. Lee (1980) shows that the real wage in England in 1800 was roughly the same to it had been in 1300. Chao (1986) shows that real wages in China at the end of eighteenth century were even lower they had been at the beginning of the first century. Lucas (1998) agues that even in the richest countries, the sustained growth phenomenon in living standards is only a few century old.

The growth of population before industrial revolution is also consistent with the predictions of the Malthusian model. The growth rate of population in Europe between the years 500 and 1500 was nearly zero (around 0.1% per year, see Figure 1). Livi-Bacci (1997) estimates the growth rate of world population from the year 1 to 1750 at 0.064% per year. The predictions of the Malthusian model are again borne out by the fluctuations in population and wages. Lee (1997) shows positive income elasticity of fertility and negative income elasticities of mortality from studying a wide range of preindustrial countries. Wrigley and Schofield (1981) find out a strong positive correlation between real wages and marriage rates in England over the period 1551 - 1801. During the period 1275 - 1801, there was a large exogenous shock, the Black Death, which reduced the population very significantly. This reduction in population was accompanied by an increase in real wage. When the population recovered, the real wages fell (see Hansen and Prescott, 2002).

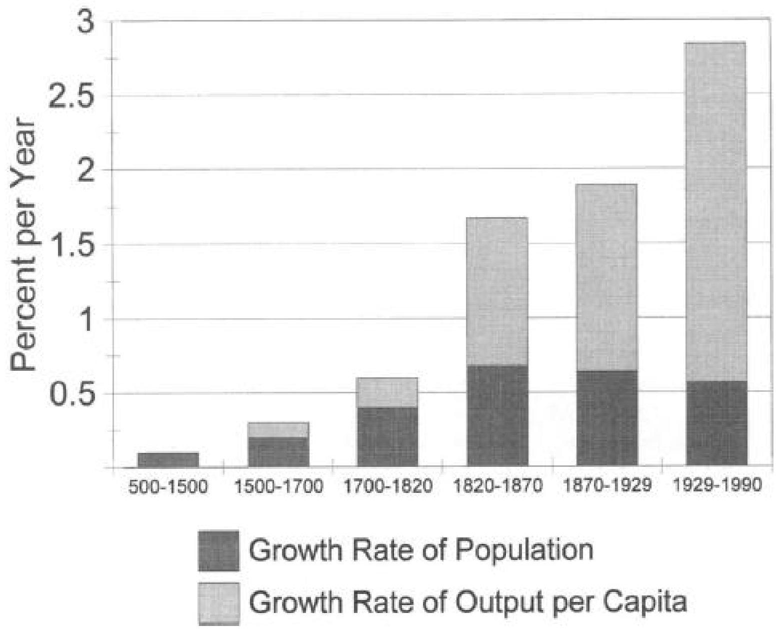

Figure 1. Output growth in Western Europe, 500 - 1990

Source: Data from 500 - 1820 are from Augus Maddison (1982) and apply to Europe as a whole. Data for 1820 - 1990 are from Maddison (1995) and apply to Western Europe[2].

The Post-Malthusian regime, which occurred between the Malthusian and Modern Growth eras, share one characteristic with each of them. Income per capita grew during this period, of course not rapidly as it would during the Modern Growth era. And at the same time, the Malthusian relationship between income per capita and population growth still holds. Rising in income was accompanied with rising in population growth rate which can be proxied by rising in fertility rate (see Figure 2). Galor (2011) argues that the simultaneity of the demographic transition across Western European countries that differed significantly in their incomes per capita suggests that the high level of income reached by these countries in the Post-Malthusian Regime played a very limited

[2] Quoted from Galor and Weil (2000)

role, if any, in the onset of the demographic transition, refuting the first testable implication of Beckerian theory (1960) which advanced the argument that the decline in fertility was a by-product of the rise in income and the associated rise in the opportunity cost of raising children. More precisely, the rise in income induced a fertility decline because the positive income effect on fertility was dominated by a negative substitution effect brought about by the rising opportunity cost of raising children.

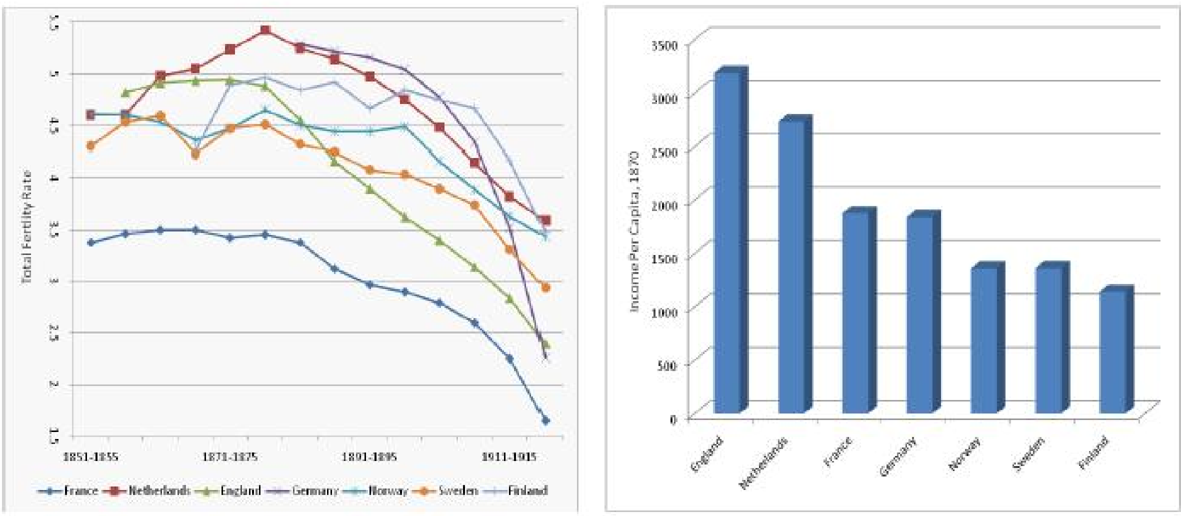

Figure 2. The Demogreaphic Transition Across Western Europe

Sources: Chesnais (1992) and Maddison (2008)[3]

The Modern Growth Regime is characterized by steady growth in both income per capita and the level of technology. In this regime there is a negative relationship between the level of income per capita and the growth rate of population. Indeed, the highest rates of population growth are observed in the poorest countries, and many rich countries have population growth rates closed to zero, or even negative.

The history of demographic transition also gives us a look, which is worth to explain and is still very limited in the literature, on the rate of fertility and infant mortality during the industrial revolution. The data for Western Europe from 1705 to 1925 shows that the decline in mortality started nearly a century prior to the decline in fertility and was initially associated with increasing in fertility rates in some countries (see Figures 2 and Figures 3).[4]

[3]Quoted from Galor (2011)

[4]One could argue that the decline in mortality was not internalized into the decisions of households who had difficulties in separating a temporary decline from a permanent one. This argument, however, is rahter implausible, given that mortality declined for nearly 140 years prior to the demographic transition. It means that around six generations did not update information about mortality rates in their immediate surroundings and kept the collective memories about mortality rates prevalent more than a century earlier.

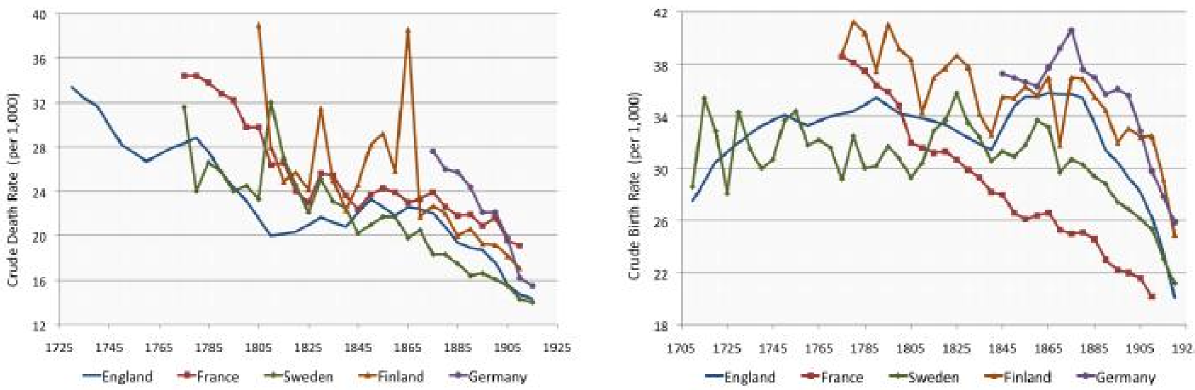

Figure 3. Mortality and Fertility across Western Europe, 1705 - 1925

Data sources: Chesnais (1992); Maddison (2008)[5]

Figure 3 shows that, except France, during the period of increasing in fertility rates from 1810 to 1875 the rates of mortality seem nondecreasing and even increasing in England and particularly in Finland. After this period the mortality rates declined again. Huck (1995) provides an evidence of rising in infant mortality rates in nine parishes in the industrial North of England from 1813 to 1836.

Parish	1813–1818	1819–1824	1825–1830	1831–1836	Total Baptisms
Walsall	177	150	156	183	7,718
Handsworth	185	185	146	131	2,483
West Bromwich	174	204	207	226	6,650
Sedgeley	157	175	183	182	9,288
Armley	164	213	216	246	3,152
Wigan	138	131	149	175	19,762
Great Harwood	135	150	152	181	2,660
Denton	172	176	143	137	3,781
Ashton	137	169	165	156	27,746
Aggregate of nine parishes	151	163	167	172	83,240
Average of nine parishes	160	173	169	180	

Table 1. Infant mortality rates after adjustment for under registration by parish (rate per thousand baptisms)

Source: Microfilms of original parish registers from the Church of Latter Day Saints Genealogical Library[6]

[5] Quoted from Galor (2005)
[6] Quoted from Huck (1993)

There is evidence for the impact of industrial revolution on environmental quality. In fact, the industrial revolution of the mid-19th century introduced new sources of air and water pollution which might affected considerably the surviving probability of infants. So, the evolutions of fertility and infant mortality during the industrial revolution need an explanation with a considering the impacts of environmental quality and education on infant mortality and on fertility behavior of households. To my best knowledge, there have been no published paper to take into account all these features in a unified model for demographic transition and economic growth. This present paper tries to make an effort to fill this big gap from the literature.

2. The Basic structure of the model

We consider an overlapping generations economy in which each agent lives two periods, say childhood and adulthood. In every period the economy produces a single homogeneous final good, using natural resource and human capital as inputs. The supply of natural resource is from nature. Using natural resource implies polluting environment. The supply of human capital is determined by agents' decisions in the previous period regarding the number and quality of their children.

2.1. Environmental Dynamic and Environmental Kuznet Curve

Environmental Dynamic

We assume that the lowest level of environmental quality is zero. The environment can regenerate itself during each period t. The regeneration capacity of environment in period t, $\Omega(E_{t-1})$, depends on the environmental quality from the previous period, i.e. E_{t-1}, where $\Omega(0) = 0$, $\Omega'(E) > 0$ for $E < E^*$, $\Omega'(E) < 0$ for $E > E^*$ and $\Omega''(E) < 0$. E^* is the environmental quality that gives us the highest level of environmental regeneration. We also assume that production process degrades the environmental quality and pollution P_t plays a role as an input of production. Such the kind dynamic of environment was first explored in "*The Entropy Law and the Economic Process*" by Georgescu-Rogen (1971). The arguments for this dynamic base on biophysical laws as the first law of thermodynamics (law of material or energy conservation) and the second law of thermodynamics (law of entropy). This stream of nonlinear dynamic of environment is then used in many papers such as Georgescu-Rogen (1975), Daly (1987, 1992), Tahvonen and Kuuluvainen (1991), Bovenberg and Smulders (1995), Smulders (1995, 2000), Fullerton and Kim (2008), etc. The environmental quality (E) evolves over time according to the following function

$$E_{t+1} = E_t + \Omega(E_t) - P_{t+1}$$

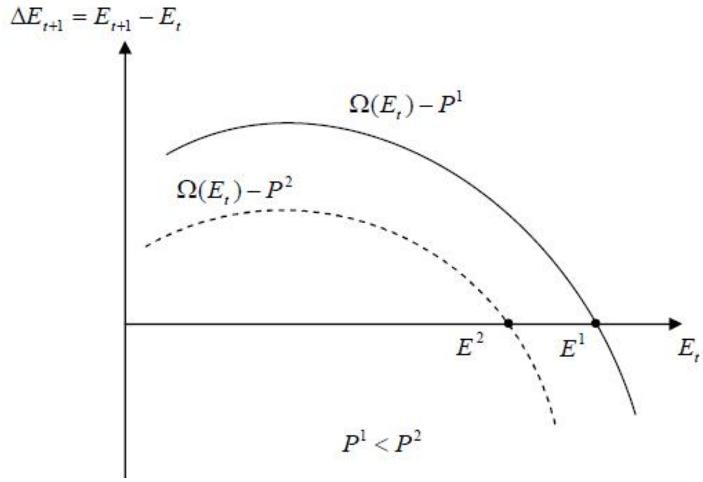

Figure 4. The regeneration of the environment

For a given level of pollution P, the environmental quality converges to a quasi-steady state $\bar{E}(P)$ that satisfies $\Omega(\bar{E}(P)) - P = 0$.

Environmental Kuznet Curve

The Environmental Kuznets Curve (EKC) hypothesis postulates an inverted-U-shaped relationship between income per capita and pollution level. A sizeable literature on EKC has grown dramatically in recent period. Two main possible explanations for this EKC are: (i) the progress of economic development, from clean agrarian economy to polluting industrial economy to clean service economy; (ii) tendency of people with higher income having higher preference for environmental quality, etc (see Dinda 2004). The dynamic of environmental quality in the present paper falls in the first explanation strain.

2.2. Production of final output

Production occurs likely according to the Constant Elasticity of Substitution that is subject to endogenous technological progress. We assume that when the level of technology is low ($0 < A_t < 1$) then the production is pollution-augmenting technological change and when the level of technology is high ($A_t \geq 1$) then the production is human capital-augmenting technological change. This assumption fits to real observations from history to present and from across countries. Indeed, for advanced economies high technology seems to be associated more with human capital while for less advanced economies technology seems to be associated more with pollution. The output produced at time t, Y_t, is

$$Y_t = \begin{cases} a(E_t)\left[\alpha H_t^{\rho} + (1-\alpha)(A_t P_t)^{\rho}\right]^{1/\rho} \; if \; 0 < A_t < 1 \\ a(E_t)\left[\alpha (A_t H_t)^{\rho} + (1-\alpha) P_t^{\rho}\right]^{1/\rho} \; if \; A_t \geq 1 \end{cases}$$

where H_t is aggregate human capital of the economy, P_t is the aggregate pollution level playing a role as an input for production, A_t is the level of technology, $a(E_t)$ is total factor productivity depending on environmental quality with $a'(E_t) > 0$, $a''(E_t) < 0$. In each period t, the producing firms choose inputs H_t and P_t to maximize their profits. Suppose that there are no property rights over natural resource. The return to pollution is therefore zero, and the return per efficiency unit of human capital is therefore equal to its average product.

$$w_t = w(A_t, E_t, h_t, p_t) = \begin{cases} a(E_t)\left[\alpha + (1-\alpha)(A_t \frac{p_t}{h_t})^\rho\right]^{1/\rho} & if\ 0 < A_t < 1 \\ a(E_t)\left[\alpha A_t^\rho + (1-\alpha)(\frac{p_t}{h_t})^\rho\right]^{1/\rho} & if\ A_t \geq 1 \end{cases} \tag{1}$$

where $p_t = P_t/L_t$ is the amount of pollution (natural resources) per worker at time t; and $w_A(A_t, E_t, h_t, p_t) > 0$, $w_E(A_t, E_t, h_t, p_t) > 0$, $w_h(A_t, E_t, h_t, p_t) < 0$.

This production form allows us to capture the inverted-U-shaped of EKC. Indeed, the first-order condition with respect to P_t gives us

$$P_t = \begin{cases} \left(\frac{(1-\alpha)A_t^\rho a(E_t)}{a'(E_t)\left[\alpha H_t^\rho + (1-\alpha)(A_t P_t)^\rho\right]}\right)^{\frac{1}{1-\rho}} & if\ 0 < A_t < 1 \\ \left(\frac{(1-\alpha)a(E_t)}{a'(E_t)\left[\alpha (A_t H_t)^\rho + (1-\alpha)P_t^\rho\right]}\right)^{\frac{1}{1-\rho}} & if\ A_t \geq 1 \end{cases} \tag{2}$$

Lemma 1: *Given $(A_t, H_t, E_{t-1}, \alpha, \rho)$, in any period t there exists a unique and strictly positive optimal level of pollution $P_t > 0$*

$$P_t = P(A_t, H_t, E_{t-1}, \alpha, \rho)$$

where $\frac{\partial P_t}{\partial A_t} > 0$ for $0 < A_t < 1$, $\frac{\partial P_t}{\partial A_t} < 0$ for $A_t \geq 1$, and $\frac{\partial P_t}{\partial H_t} < 0$.

Proof:
For the case $0 < A_t < 1$,

$$P_t = \left(\frac{(1-\alpha)A_t^\rho a(E_t)}{a'(E_t)\left[\alpha H_t^\rho + (1-\alpha)(A_t P_t)^\rho\right]}\right)^{\frac{1}{1-\rho}} \tag{3}$$

Let $G_1 = G_1(A_t, H_t, P_t) = \left(\frac{(1-\alpha)A_t^\rho a(E_t)}{a'(E_t)\left[\alpha H_t^\rho + (1-\alpha)(A_t P_t)^\rho\right]}\right)^{\frac{1}{1-\rho}}$ we have

$$\underset{P_t \to 0}{lim} G_1 = \left(\frac{(1-\alpha)A_t^\rho a(E_{t-1} + \Omega(E_{t-1}))}{a'(E_{t-1} + \Omega(E_{t-1}))\alpha H_t^\rho}\right)^{\frac{1}{1-\rho}} > 0$$

$$\underset{P_t \to E_{t-1} + \Omega(E_{t-1})}{lim} G_1 = \left(\frac{(1-\alpha)A_t^\rho a(0)}{a'(0)\left[\alpha H_t^\rho + (1-\alpha)(A_t P_t)^\rho\right]}\right)^{\frac{1}{1-\rho}} = 0$$

and

$$\frac{\partial G_1}{\partial P_t} = \frac{(\alpha - 1)A_t^\rho G_1^\rho}{1-\rho}.\Phi_1 < 0$$

where $\Phi_1 = \frac{a'(E_t)^2\left[\alpha H_t^\rho + (1-\alpha)(A_t P_t)^\rho\right] + a(E_t)\left(-a''(E_t)\left[\alpha H_t^\rho + (1-\alpha)(A_t P_t)^\rho\right] + a'(E_t)\rho(1-\alpha)A_t^\rho P_t^{\rho-1}\right)}{a'(E_t)^2\left[\alpha H_t^\rho + (1-\alpha)(A_t P_t)^\rho\right]^2} > 0$.
So for the case $0 < A_t < 1$, there exists a unique strictly positive level of optimal pollution which is the intersection between the 45° line and the curve G_1 as presented in the Figure 5. Now, we prove that $\frac{\partial P}{\partial A_t} > 0$ and $\frac{\partial P}{\partial H_t} < 0$. Indeed, we fix H_t and P_t and vary A_t, it is obvious that the line G_1 rotates clockwise as A_t increases up to $\tilde{A}_t$. Consequently, the intersection between the curve G_1 and the 45° line will be at $(\tilde{P}_t, \tilde{P}_t)$ where $\tilde{P}_t > P_t$ (see Figure 5). Hence, $\frac{\partial P}{\partial A_t} > 0$. We can use the analogous arguments to show that $\frac{\partial P}{\partial H_t} < 0$.

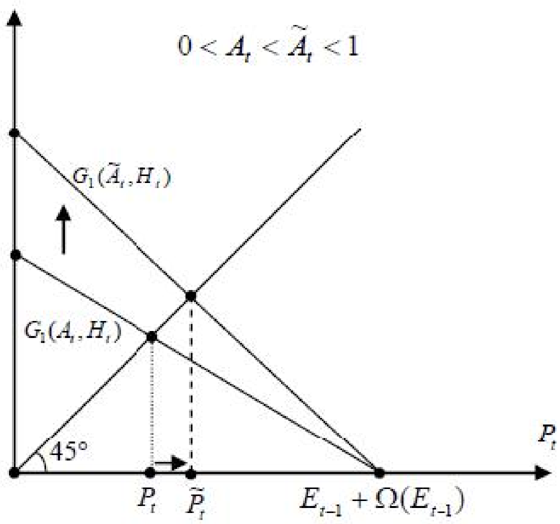

Figure 5.

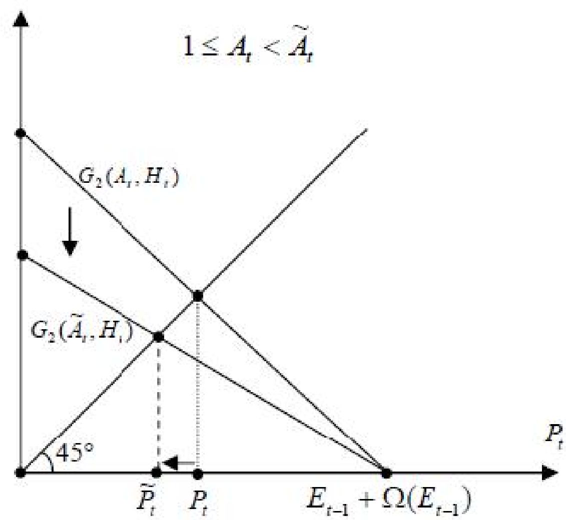

Figure 6.

Similarly, for the case $A_t \geq 1$,

$$P_t = \left(\frac{(1-\alpha)a(E_t)}{a'(E_t)\left[\alpha(A_t H_t)^\rho + (1-\alpha)P_t^\rho\right]}\right)^{\frac{1}{1-\rho}} \tag{4}$$

Let $G_2 = \left(\frac{(1-\alpha)a(E_t)}{a'(E_t)\left[\alpha(A_t H_t)^\rho + (1-\alpha)P_t^\rho\right]}\right)^{\frac{1}{1-\rho}}$ we have

$$\lim_{P_t \to 0} G_2 = \left(\frac{(1-\alpha)a(E_{t-1} + \Omega(E_{t-1}))}{\alpha a'(E_{t-1} + \Omega(E_{t-1}))(A_t H_t)^\rho}\right)^{\frac{1}{1-\rho}} > 0$$

$$\lim_{P_t \to E_{t-1} + \Omega(E_{t-1})} G_2 = \left(\frac{(1-\alpha)a(0)}{a'(0)\left[\alpha(A_t H_t)^\rho + (1-\alpha)P_t^\rho\right]}\right)^{\frac{1}{1-\rho}} = 0$$

and

$$\frac{\partial G_2}{\partial P_t} = \frac{\alpha - 1}{1-\rho} G_2^\rho \Phi_2 < 0$$

where $\Phi_2 = \frac{a'(E_t)^2\left[\alpha(A_tH_t)^\rho+(1-\alpha)P_t^\rho\right]+a(E_t)\left(-a''(E_t)\left[\alpha(A_tH_t)^\rho+(1-\alpha)P_t^\rho\right]+a'(E_t)\rho(1-\alpha)P_t^{\rho-1}\right)}{a'(E_t)^2\left[\alpha(A_tH_t)^\rho+(1-\alpha)P_t^\rho\right]^2} > 0$.
So for the case $A_t \geq 1$, there exists a unique strictly positive level of optimal pollution which is the intersection between the 45° line and the curve G_2 as presented in the Figure 6. Now, we prove that $\frac{\partial P}{\partial A_t} < 0$ and $\frac{\partial P}{\partial H_t} < 0$. Indeed, we fix H_t and P_t and vary A_t, it is obvious that the line G_2 rotates counter-clockwise as A_t increases up to $\tilde{A}_t$. Consequently, the intersection between the curve G_1 and the 45° line will be at $(\tilde{P}_t, \tilde{P}_t)$ where $\tilde{P}_t < P_t$ (see Figure 6). Hence, $\frac{\partial P}{\partial A_t} > 0$. We can use the analogous arguments to show that $\frac{\partial P}{\partial H_t} < 0$. QED.

So there exists an inverted-U-shape between pollution and level of technology. This result supports for the explanation of ECK curve that the progress of economic development, from clean agrarian economy to polluting industrial economy to clean service economy.

2.3. Households

2.3.1. Preferences and budget constraints of households

We follow the standard model of household fertility behavior in Becker (1960). The household chooses the number of children and their quality in the face of a constraint on the total amount of time that can be used to childrearing and labor-market activities. Following Galor and Weil (2000) we also further assume that the only input required to give birth and to produce both child quantity and child quality is time.

In each period t a genreration that consists of L_t identical agents joins the labor force. Each agent t has a single parent and lives for two periods. In the first period of life (say childhood period), $t-1$, the agent consumes a fraction of his parent's time. In the second period of life (say parental period), t, he is endowed with one unit of time, which he allocates between child-rearing and labor force participation. He chooses the optimal mixture of quantity and quality of surviving children and supply his remaining time in the labor market to consume his wages. The preferences of the agent born in period $t-1$ are defined over the number and quality of his surviving children, n_t and h_{t+1}, respectively, as well as from his consumption in period t, c_t. They are represented by the following utility function

$$u_t = \gamma\left[\ln n_t + \ln h_{t+1}\right] + (1-\gamma)\ln c_t \tag{5}$$

Let $\frac{\phi^b}{\pi_t} + \phi^q + \phi^e e_{t+1}$ be the time cost for an agent t of raising a surviving child with a level of education e_{t+1}. That is, ϕ^b is the cost in time to give a birth, ϕ^q is the cost in time to raise physically a surviving child, regardless of quality, and ϕ^e is the cost in time required for each unit of education for each child. $\pi_t = \pi(e_t, E_t)$ is the surviving probability of children when they are born. This probability depends positively on both environmental quality E_t and education level of the parent e_t. We define the potential income of the agent t as the amount $z_t = w_t h_t$ he would earn if he devoted his entire time endowment (which is normalized by 1) to the labor market, w_t is return to per efficient unit of human capital which is defined by eqution (1) and h_t is the human capital of the agent. The potential income is divided between expenditure on child-rearing (giving birth, quantity as well as quality) with an opportunity cost of $w_t h_t[\frac{\phi^b}{\pi_t} + \phi^q + \phi^e e_{t+1}]$ per surviving child, and consumption c_t. Hence, the agent born at date $t-1$ faces at date t the following budget constraint

$$w_t h_t n_t[\frac{\phi^b}{\pi_t} + \phi^q + \phi^e e_{t+1}] + c_t \leq w_t h_t \tag{6}$$

2.3.2. Human capital formation

The human capital of an agent is determined by his education level (which is provided by his parent) and by the technogical progress. As in Galor and Weil (2000), we assume that human capital of children born at date t, h_{t+1}, is an increasing function of their education e_{t+1}, and a decreasing function of the rate of technological progress from period t to period $t+1$, $g_{t+1} = \frac{A_{t+1}-A_t}{A_t}$. The higher the children's education, the smaller the adverse effect of technological progress.

$$h_{t+1} = h(e_{t+1}, g_{t+1}) \tag{7}$$

where $h_{t+1} = h(e_{t+1}, g_{t+1}) > 0$, $h_e(e,g) > 0$, $h_{ee}(e,g) < 0$, $h_g(e,g) < 0$, $h_{gg}(e,g) < 0$, $h_{eg}(e,g) > 0$ $\forall(e_{t+1}, g_{t+1}) \geq 0$ and $\underset{g_{t+1}\to\infty}{lim} h(0, g_{t+1}) = 0$. Hence, the human capital level of children is an increasing and strictly concave function of education, and a decreasing and strictly convex function of technological progress.[7] Education lessens the obsolescence of human capital in a changing technology. The marginal productivity of parental investment in a child's human capital increases in a more rapidly changing technological environment, i.e. $h_{eg}(e,g) > 0$.

2.3.3. Optimization

Each agent of generation t chooses the number and quality of his surviving children, and therefore the human capital of children and his own consumption, so as to maximize his utility. Substituting (6) and (7) into (5), the optimization problem of the agent t is

$$(n_t, e_{t+1}) \in argmax\, \gamma\,[\ln n_t + \ln h(e_{t+1}, g_{t+1})] + (1-\gamma)\ln w_t h_t(1 - n_t[\frac{\phi^b}{\pi_t} + \phi^q + \phi^e e_{t+1}])$$

subject to

$$w_t h_t(1 - n_t[\frac{\phi^b}{\pi_t} + \phi^q + \phi^e e_{t+1}]) \geq 0$$

$$n_t \geq 0,\ e_{t+1} \geq 0$$

For solving this optimization problem, see the Appendix. The first-order condition with respect to n_t gives us

[7]Strictly convexity with respect to g_{t+1} is not essential. This property just ensures that the level of human capital will not be zero at high rates of technological progress.

$$n_t = \frac{\gamma}{\frac{\phi^b}{\pi_t} + \phi^q + \phi^e e_{t+1}} \tag{8}$$

So the optimization with respect to n_t implies that the time spent raising children by agent t is γ, whereas $1-\gamma$ is devoted for labor market, and hence the optimal consumption is always strictly positive, $c_t > 0$.

It is shown in Galor and Weil (2000) that, an increase in income does not change the division of childrearing time between quality and quantity. This argument still holds in our context, however, we extent the argument to capture also the time for giving birth. What *do* affect the division between time spent on giving births, time spent on quality and time spent on quantity of children are the rate of technological progress, which changes the return to education, and the surviving probability of infant children, which changes the fertility behavior of the agent.

The optimization with respect to e_{t+1} gives us the implicit functional relationships between e_{t+1} and g_{t+1}, π_t are given by

$$G(e_{t+1}, g_{t+1}, \pi_t) = h_e(e_{t+1}, g_{t+1})[\frac{\phi^b}{\pi_t} + \phi^q + \phi^e e_{t+1}] - \phi^e h(e_{t+1}, g_{t+1}) \begin{cases} = 0 \; if \; e_{t+1} > 0 \\ \leq 0 \; if \; e_{t+1} = 0 \end{cases} \tag{9}$$

where $G_e(e_{t+1}, g_{t+1}, \pi_t) < 0$, $G_g(e_{t+1}, g_{t+1}, \pi_t) > 0$ and $G_\pi(e_{t+1}, g_{t+1}, \pi_t) < 0 \; \forall g_{t+1} \geq 0$ and $\forall e_{t+1} > 0$. To ensure the existence of positive level of g_{t+1} such that the chosen level of education is 0, we assume that

$$G(0, 0, \pi_t) = h_e(0,0)[\frac{\phi^b}{\pi_t} + \phi^q] - \phi^e h(0,0) < 0 \quad \forall \pi_t \in [0,1] \qquad (A1)$$

Lemma 2: *If (A1) is satisfied, then the level of education chosen by members of generation t for their children is a nondecreasing function of g_{t+1}.*

$$e_{t+1} = e(g_{t+1}, \pi_t) \begin{cases} = 0 \; if \; g_{t+1} \leq \hat{g} \\ > 0 \; if \; g_{t+1} > \hat{g} \end{cases} \qquad \forall \pi_t \in [0,1]$$

where $\hat{g} > 0$, and

$$e_g(g_{t+1}, \pi_t) > 0, \; e_\pi(g_{t+1}, \pi_t) < 0 \; \forall g_{t+1} > \hat{g}$$

Proof:

As follows from the assumptions of human capital formation function and (9), we find that $G(0, g_{t+1}, \pi_t)$ is monotonically increasing in g_{t+1}. In effect,

$$\frac{\partial G(0, g_{t+1}, \pi_t)}{\partial g_{t+1}} = h_{eg}(0, g_{t+1})[\frac{\phi^b}{\pi_t} + \phi^q] - \phi^e h_g(0, g_{t+1}) > 0$$

Furthermore, $\lim_{g_{t+1}\to+\infty} G(0, g_{t+1}, \pi_t) > 0$ whereas $(A1)$ implies that $G(0, 0, \pi_t) < 0$. Hence, there exists a unique $\hat{g} > 0$ such that $G(0, \hat{g}, \pi_t) = 0$, and therefore, as follows from (9) $e_{t+1} = 0$ for $g_{t+1} \leq \hat{g}$. Furthermore, by using implicit function theorem, it follows from (9) that e_{t+1} is a single-value function of g_{t+1} and π_t, $e_{t+1} = e(g_{t+1}, \pi_t)$ where $\partial e_{t+1}/\partial g_{t+1} = -G_g(e_{t+1}, g_{t+1}, \pi_t)/G_e(e_{t+1}, g_{t+1}, \pi_t) > 0$, $\partial e_{t+1}/\partial \pi_t = -G_\pi(e_{t+1}, g_{t+1}, \pi_t)/G_e(e_{t+1}, g_{t+1}, \pi_t) < 0$. Q.E.D.

By implicit function theorem, we find that

$$
\begin{aligned}
e_{\pi\pi}(g_{t+1}, \pi_t) = \frac{\partial(-G_\pi/G_e)}{\partial \pi_t} &= \frac{G_\pi G_{e\pi} - G_{\pi\pi}G_e}{G_e^2} \\
&= \frac{-h_e h_{ee} \phi^b(\phi^b + 2\pi_t[\phi^q + \phi^e e_{t+1}])}{\pi_t^4 G_e^2} > 0
\end{aligned}
$$

Hence, $e(g_{t+1}, \pi_t)$ is a decreasing convex function in π_t. We assume that

$$\lim_{\pi_t \to 1^-} e(g_{t+1}, \pi_t) + \pi_t e_\pi(g_{t+1}, \pi_t) > 0 \quad \forall g_{t+1} > \hat{g} \qquad (A2)$$

We can see from (9) that $e_{gg}(g_{t+1}, \pi_t)$ depends on the third derivatives of the human capital formation function, $h(e_{t+1}, g_{t+1})$. According to Galor and Weil (2000), a concave reaction of the level of education to the rate of technological progress appears plausible economically, hence we can assume that[8]

$$e_{gg}(g_{t+1}, \pi_t) < 0 \quad \forall g_{t+1} > \hat{g} \qquad (A3)$$

Substituting $e_{t+1} = e(g_{t+1}, \pi_t)$ into (8), we have

$$n_t = \frac{\gamma}{\frac{\phi^b}{\pi_t} + \phi^q + \phi^e e(g_{t+1}, \pi_t)} \tag{10}$$

From (1) and defintion of z_t, we have

$$z_t = z(e_t, g_t, p_t) = w_t h_t = \begin{cases} a(E_t)\left[\alpha h_t^\rho + (1-\alpha)(A_t p_t)^\rho\right]^{1/\rho} & if\ 0 < A_t < 1 \\ a(E_t)\left[\alpha (A_t h_t)^\rho + (1-\alpha) p_t^\rho\right]^{1/\rho} & if\ A_t \geq 1 \end{cases}$$

where $z_e(e_t, g_t, p_t) > 0$, $z_g(e_t, g_t, p_t) < 0$, and $z_p(e_t, g_t, p_t) = 0$.

The following proposition summarizes the properties of the functions $e(g_{t+1}, \pi_t)$, $n(g_{t+1}, \pi_t)$ and their significance for the evolutions in fertility and the substitution of quality for quantity of surviving children for the process of development.

[8] Galor and Weil (2000) note that, if $e(g_{t+1})$ is strictly convex we may assume that for physiological or other reasons, the maximum amount of education that a child can receive is bounded from above. In the model we ignore integer constraints on the numer of children, so that absent a constraint on the quality per child, parent might choose to have an infinitesimally small number of children with infinitely high quality. Thus the existence of integer constraints may be taken as one justification for an upper bound on level of education.

Proposition 1: *Under (A1) and (A2)*

(1) Technological progress that is expected to occur between the first and second periods of children's lives results in a decline in the fertility rate, a decline parents' chosen number of surviving children and an increase in their quality, i.e.

$$\frac{\partial n_t^b}{\partial g_{t+1}} \leq 0, \quad \frac{\partial n_t}{\partial g_{t+1}} \leq 0, \quad \text{and} \quad \frac{\partial e_{t+1}}{\partial g_{t+1}} \geq 0$$

(2) An increase in environmental quality as well as an increase in parental education level result in an increase in the surviving probability of infant, which in turn results in a decline in the fertility rate and an increase in number of surviving children, i.e.,

$$\frac{\partial n_t^b}{\partial \pi_t} < 0 \quad \text{and} \quad \frac{\partial n_t}{\partial \pi_t} > 0$$

(3) An increase in parental potential income does not change the fertility rate and number of surviving children or their quality, i.e.,

$$\frac{\partial n_t^b}{\partial z_t} = 0, \quad \frac{\partial n_t}{\partial z_t} = 0, \quad \text{and} \quad \frac{\partial e_{t+1}}{\partial z_t} = 0$$

Proof:

(1) Follows directly from the result in Lemma 2 that $e_g(g_{t+1}, \pi_t) > 0 \, \forall g_{t+1} > \hat{g}$ and $e_g(g_{t+1}, \pi_t) = 0 \, \forall g_{t+1} < \hat{g}$, and from equation (10)

(2) Follows directly from the result in Lemma 2 that $e_\pi(g_{t+1}, \pi_t) < 0 \, \forall g_{t+1} > \hat{g}$ and $e_\pi(g_{t+1}, \pi_t) = 0 \, \forall g_{t+1} < \hat{g}$, and from equation (10) we have

$$\frac{\partial n_t}{\partial \pi_t} > 0$$

The fertility rate in period t is the number of children born per person, n_t^b

$$n_t^b = \frac{n_t}{\pi_t} = \frac{\gamma}{\phi^b + \pi_t[\phi^q + \phi^e e(g_{t+1}, \pi_t)]} \tag{11}$$

We have

$$\frac{\partial n_t^b}{\partial \pi_t} = -\frac{\gamma[\phi^q + \phi^e(e(g_{t+1}, \pi_t) + \pi_t e_\pi(g_{t+1}, \pi_t))]}{(\phi^b + \pi_t[\phi^q + \phi^e e(g_{t+1}, \pi_t)])^2}$$

We will show that $\Lambda = e(g_{t+1}, \pi_t) + \pi_t e_\pi(g_{t+1}, \pi_t) > 0$. In effect,

$$\frac{\partial \Lambda}{\partial \pi_t} = 2e_\pi(g_{t+1}, \pi_t) + \pi_t e_{\pi\pi}(g_{t+1}, \pi_t)$$

Using (9) and applying implicit function theorem we have

$$\frac{\partial \Lambda}{\partial \pi_t} = -2\frac{G_\pi}{G_e} - \pi_t \frac{G_{\pi\pi}G_e - G_\pi G_{e\pi}}{G_e^2}$$

$$= \frac{\pi_t G_\pi G_{e\pi} - G_e(2G_\pi - \pi_t G_{\pi\pi})}{G_e^2} = \frac{(\phi^b)^2 h_e h_{ee}}{\pi_t^4} < 0$$

Hence, Λ is decresing in π_t, then we have

$$\Lambda > \lim_{\pi_t \to 1^-} e(g_{t+1}, \pi_t) + \pi_t e_\pi(g_{t+1}, \pi_t) > 0 \quad \forall \pi_t \in [0, 1)$$

Therefore,

$$\frac{\partial n_t^b}{\partial \pi_t} < 0$$

(3) Follows directly from the equations (8) and (9). Q.E.D.

2.4. Technological progress

As in Galor and Weil (2000), we assume that technological progress taking place between periods t and $t+1$ depends on the average education among the working generation in period t, e_t, and the population size of the working generation in period t, L_t.

$$g_{t+1} = \frac{A_{t+1} - A_t}{A_t} = g(e_t, L_t) \tag{12}$$

where for $L_t \gg 0$ and $e_t \geq 0$, $g(0, L_t) > 0$, $g_i(e_t, L_t) > 0$, and $g_{ii}(e_t, L_t) < 0$ with $i \in \{e_t, L_t\}$.[9]

Hence, when the population size is large enough then the rate of technological progress between period t and period $t+1$ is always positive, even if labor quality is zero, increasing, and strictly concave function of the population size and level of education of working generation at time t.

In this stage, in order to simplify the exposition, the dynamical system is analyzed initially under the assumption that an increase in the size of population has no effect on technological progress, i.e., we initially assume that

$$g_L(e_t, L_t) = 0 \quad \forall L_t > 0 \qquad (A4)$$

In the later stages of the analysis, the effect of population size on the relationship between technological progress and the level of education as specified in (12) is fully taken into account in the analysis.

[9] As in Galor and Weil (2000), we also assume that for a sufficient small population $L_t > 0$ then $g(0, L_t) \geq 0$, $g_i(e_t, L_t) \geq 0$ for all t, and $g(0, L_t) > 0$, $g_i(e_t, L_t) > 0$ for some t. These assumptions ensure that in the early stages of development the economy is indeed in a Malthusian steady state. And if technological progress occurred in every time period at a pace that increased with the size of population, the growth rate of output per capita would always be positive, despite the adjustment in the size of population.

2.5. The Evolution of Population, Technology and Pollution

The size of the working population in the period $t+1$, L_{t+1}, is

$$L_{t+1} = n_t L_t \tag{13}$$

where L_t is the size of the working population in period t, n_t is the number of children per person, and hence $n_t - 1$ is the growth rate of population. The size of the working population in period 0 is historically given at a level L_0.

The level of technology in period $t+1$, A_{t+1}, is derived from equation (12) as

$$A_{t+1} = (1 + g_{t+1})A_t \tag{14}$$

where the level of technology in period 0 is historically given at a level A_0.

The evolution of environmental quality depends on the evolution of the technology level,

$$E_{t+1} = \begin{cases} E_t + \Omega(E_t) - \left(\frac{(1-\alpha)A^{\rho}_{t+1}a(E_{t+1})}{a'(E_{t+1})\left[\alpha H^{\rho}_{t+1}+(1-\alpha)(A_{t+1}P_{t+1})^{\rho}\right]}\right)^{\frac{1}{1-\rho}} & if\ 0 < A_{t+1} < 1 \\ E_t + \Omega(E_t) - \left(\frac{(1-\alpha)a(E_{t+1})}{a'(E_{t+1})\left[\alpha(A_{t+1}H_{t+1})^{\rho}+(1-\alpha)P^{\rho}_{t+1}\right]}\right)^{\frac{1}{1-\rho}} & if\ A_{t+1} \geq 1 \end{cases} \tag{15}$$

where P_{t+1} is implictly defined (see equation (2)) by

$$P_{t+1} = \begin{cases} \left(\frac{(1-\alpha)A^{\rho}_{t+1}a(E_{t+1})}{a'(E_{t+1})\left[\alpha H^{\rho}_{t+1}+(1-\alpha)(A_{t+1}P_{t+1})^{\rho}\right]}\right)^{\frac{1}{1-\rho}} & if\ 0 < A_{t+1} < 1 \\ \left(\frac{(1-\alpha)a(E_{t+1})}{a'(E_{t+1})\left[\alpha(A_{t+1}H_{t+1})^{\rho}+(1-\alpha)P^{\rho}_{t+1}\right]}\right)^{\frac{1}{1-\rho}} & if\ A_{t+1} \geq 1 \end{cases} \tag{16}$$

3. The Dynamic system

The development of the economy is characterized by the evolution of output per capita, technological level, education per worker, human capital, and environmental quality. The evolution of the economy is fully determined by a sequence $\{e_t, g_t, E_t, L_t\}_{t=0}^{\infty}$ that satisfies (13)-(16) and Lemma 2 in every period t.

3.1. The evolution of Technology and Education

The evolution of technology and education, given $(A4)$, is characterized by the sequence $\{g_t, e_t\}_{t=0}^{\infty}$ that satisfies in every period t the equations $g_{t+1} = g(e_t, L_t)$ and $e_{t+1} = e(g_{t+1}, \pi_t)$. In the light of the properties of the functions $e(g_{t+1}, \pi_t)$ and $g(e_t, L_t)$ given in Lemma 2, $(A3)$, $(A4)$, and (12), it follows that if the size of population plays a role in technological progress, this dynamical subsystem is characterized qualitatively by three different configurations, which are depicted in Figures 7 - 9. The economy shifts endogenously from one configuration to another as the size of population increases and the curve $g(e_t, L_t)$ shifts upward to account for the effect of an increase in population.

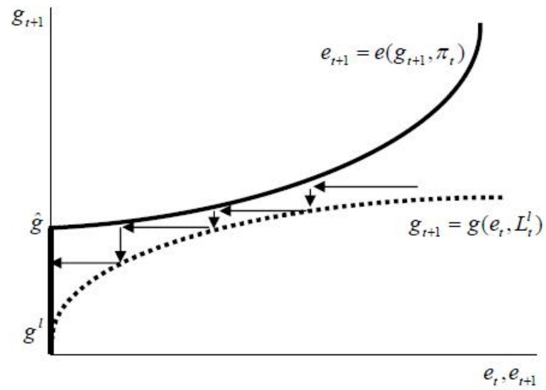

Figure 7. The evolution of technology and education for a small population

Figure 7 describes the evolutions of education e_t and the rate of technological progress g_{t+1} for a constant small population L_t^l and constant surving infant probability π_t. The curve $g_{t+1} = g(e_t, L_t^l)$ describes the effect of education on the rate of technological progress as presented in (12). The curve $e_{t+1} = e(g_{t+1}, \pi_t)$ describes the effect of the expected rate of technological progress on optimal education choices derived in Lemma 2. The intersection between the two curves is the globally stable steady state equilibrium $(0, g^l)$. This figure implies that, in early stage of development, the economy is in the vicinity of this steady state in which education is zero and the rate of technological progress is very low. It can also be predicted that the infant mortality rate would be high due to the low education of their parents. And hence from Proposition 1, the fertility rate would be high but the surviving children would be low. This prediction is consistent with the historical fact of thousand of years of Malthusian stagnation.

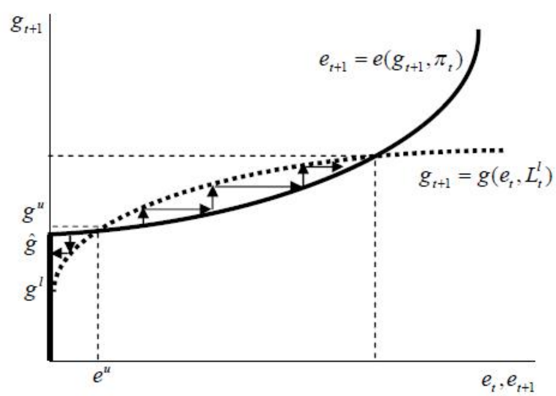

Figure 8. The evolution of technology and education for a moderate population

Figure 8 describes the evolution of education and the technological progress when the population has grown to reach a moderate size. The system now is characterized by multiple steady state equilibria. The steady state equilibria $(0, g^l)$ and (e^h, g^h) are locally stable, whereas (e^u, g^u) is unstable. Given the initial conditions with low education and low rate of the technological progress, in the absence of large shocks the economy remains in the vicinity of the low steady state equilibrium $(0, g^l)$ in which education is still zero but the rate of technological progress is moderate. This figure also suggests an idea to interpret the fact acrossing countries that some countries have high education level as well as high growth rate of technological progress associated with low fertility rate (and hence low population growth), while some other countries have low education level as well as low growth rate of technological progress associated with high fertility rate.

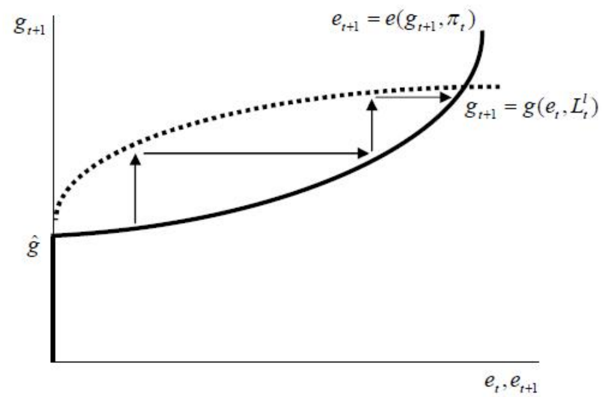

Figure 9. The evolution of technology and education for a large population

Figure 9 describes the evolution of education and the rate of technological progress when the population grows to a high level. The system is characterized by a unique globally stable steady state equilibrium (e^h, g^h). So in the mature stage of development, the economy converges monotonically to this steady state with high levels of education and technological progress.

3.2. The evolution of infant surviving probability along with technological progress

It is worthy to investigate the evolution of the infant surviving probability to understand the oscillation of fertility rate over time and understand why in the history the infant mortality rate is high. We have

$$\pi_t = \pi(e_t, E_t) = \pi(e(g_t, \pi_{t-1}), E_t)$$

$$\begin{aligned} \frac{\partial \pi_t}{\partial g_t} &= \frac{\partial \pi(e_t, E_t)}{\partial e_t}\frac{\partial e(g_t, \pi_{t-1})}{\partial g_t} + \frac{\partial \pi(e_t, E_t)}{\partial E_t}.\frac{\partial E_t}{\partial P_t}.\frac{\partial P_t}{\partial A_t}.\frac{\partial A_t}{\partial g_t} \\ &= \frac{\partial \pi(e_t, E_t)}{\partial e_t}\frac{\partial e(g_t, \pi_{t-1})}{\partial g_t} - \frac{\partial \pi(e_t, E_t)}{\partial E_t}\frac{\partial P_t}{\partial A_t}A_{t-1} \end{aligned} \tag{17}$$

3.2.1. Small population

We assume that for the very early stage of development, population is small, and the technology progress is very low, which is characterized by $g_t < \hat{g}$ and $A_t \ll 1$, then $\frac{\partial e(g_t,\pi_{t-1})}{\partial g_t} = 0$ and hence,

$$\frac{\partial \pi_t}{\partial g_t} = -\frac{\partial \pi(e_t, E_t)}{\partial E_t}\frac{\partial P_t}{\partial A_t}A_{t-1} < 0$$

This implies that in the very early stage of development the surviving probability of children is a decreasing function of technological progress. It is that when technoligical progress was low enough that agents were not invested in education by their parent from previous period, $t-1$, while technological progress in period t makes the enviroment more polluted. As the result, the pollution suffers infants without compensation from education of their parent. In this case, as showed in the Proposition 1, the number of children born per agent increases and the number of surviving children per agent decreases.

3.2.2. Moderate population

For moderate population (Figure 8) there are two possibilities. If the rate of technological progress is still low enough, the economy converges to very low and stable steady state which is characterized by $e = 0$ and $g \leq \hat{g}$. In this case the evolution of infant surviving probability is similar to the one in the case of small population.

If the rate of technological progress is large enough, $g > \hat{g}$, we consider two cases of technological level

(i) If $A_t \geq 1$ then $\frac{\partial P_t}{\partial A_t} < 0$, and hence

$$\frac{\partial \pi_t}{\partial g_t} = \frac{\partial \pi(e_t, E_t)}{\partial e_t}\frac{\partial e(g_t, \pi_{t-1})}{\partial g_t} - \frac{\partial \pi(e_t, E_t)}{\partial E_t}\frac{\partial P_t}{\partial A_t}A_{t-1} > 0$$

In this case there are two effects, which reinforce each other, on surviving probability of infant. Firstly, the expected technological progress in period t is large enough makes agents from period $t-1$ having incentive to invest in education for their children. As a result, the more education the agents in period t are, the larger the surviving probability of their children is. Secondly, technological progress makes the economy less polluted, and hence the better environment quality has positive effect on the surviving probability of children.

(ii) If $A_t < 1$ then $\frac{\partial P_t}{\partial A_t} > 0$,

In this case there are two effects, which offset each other, on surviving probability of infant. Firstly, the expected technological progress in period t is large enough makes agents from period $t-1$ having incentive to invest in education for their children. As a result, the more education the agents in period t are, the larger the surviving probability of their children is. Secondly, technological progress makes the economy more polluted, and hence the worse environment quality has negative effect on the surviving probability of children. Hence, the impact of technological progress on surviving probability of children depends on which effect (education or environment) dominates the other.

3.2.3. Large population

For a large population, as shown in figure 9, the economy converges to the steady state equilibrium (e^h, g^h) where $g^h > \hat{g}$ and $e^h > 0$. It is reasonable to assume in this case that $A_t \geq 1$, hence $\frac{\partial P_t}{\partial A_t} < 0$ and then

$$\frac{\partial \pi_t}{\partial g_t} = \frac{\partial \pi(e_t, E_t)}{\partial e_t}\frac{\partial e(g_t, \pi_{t-1})}{\partial g_t} - \frac{\partial \pi(e_t, E_t)}{\partial E_t}\frac{\partial P_t}{\partial A_t}A_{t-1} > 0$$

So this case is exactly similar to the case in (1) of subsection 3.2.2 in which two effects (education and environment) reinforce each other to raise the surviving probability of children along with the increase in the growth rate of technological progress.

3.3. The Dynamics of surviving probability of children

Now we investigate the dynamics of surviving probability of children to study its oscillation over time. We have

$$\frac{\partial \pi_t}{\partial \pi_{t-1}} = \pi_e(e(g_t, \pi_{t-1}), E_t).e_\pi(g_t, \pi_{t-1}) < 0$$

We know that for any given E_t, π_t is bounded in the interval $[0, 1]$, so the explosiveness in the dynamic of the infant surviving probability is impossible. So it seems reasonable to assume that the slope of the curve $\pi(e(g_t, \pi_{t-1}), E_t)$ with respect to π_{t-1} has absolute value less than one, which implies that $\frac{\partial \pi_t}{\partial \pi_{t-1}} > -1$. So for any given environmental quality $E_t = E\ \forall t$, there would be a convergence of the infant surviving probability which display an oscillation (see figure 10).

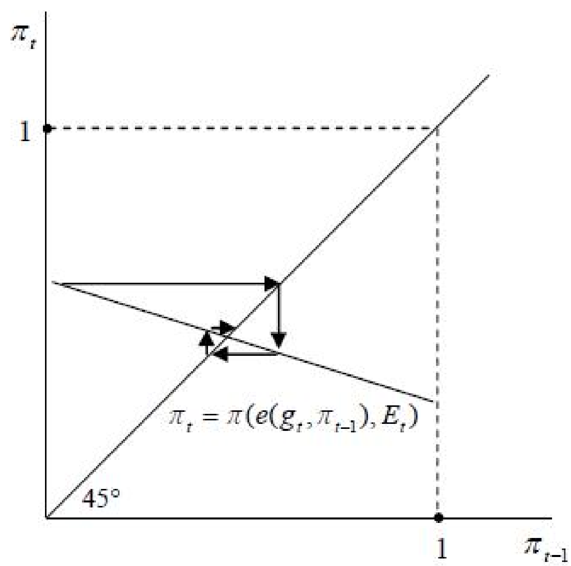

Figure 10. The convergence of infant surviving probability

Economically, the higher (lower) the infant surviving probability in period t (i.e. π_t) is, the higher (lower) the number of surviving children is also (as stated in Proposition

1). Then, for a given potential income of the household, the education invested for each child would be lower (higher), resulting in a negative (positive) effect on the infant surviving probability in the next period, π_{t+1}.

3.3. Global Dynamic

The global dynamic of the economy is characterized by $\{n_t, e_{t+1}, h_{t+1}, g_{t+1}, A_{t+1}, L_{t+1}, E_{t+1}, P_{t+1}, \pi_t\}_{t=0}^{\infty}$, for given initial conditions e_0, A_0, L_0, E_0, which satisfies the following system of nine equations

$$n_t = \frac{\gamma}{\frac{\phi^b}{\pi_t} + \phi^q + \phi^e e_{t+1}}$$

$$\pi_t = \pi(e_t, E_t)$$

$$h_e(e_{t+1}, g_{t+1})[\frac{\phi^b}{\pi_t} + \phi^q + \phi^e e_{t+1}] - \phi^e h(e_{t+1}, g_{t+1}) \begin{cases} = 0 \; if \; e_{t+1} > 0 \\ \leq 0 \; if \; e_{t+1} = 0 \end{cases}$$

$$h_{t+1} = h(e_{t+1}, g_{t+1})$$

$$g_{t+1} = g(e_t, L_t)$$

$$A_{t+1} = (1 + g_{t+1})A_t$$

$$L_{t+1} = n_t L_t$$

$$E_{t+1} = E_t + \Omega(E_t) - P_{t+1}$$

$$P_{t+1} = \begin{cases} \left(\frac{(1-\alpha)A_{t+1}^{\rho} a(E_{t+1})}{a'(E_{t+1})\left[\alpha H_{t+1}^{\rho} + (1-\alpha)(A_{t+1}P_{t+1})^{\rho}\right]}\right)^{\frac{1}{1-\rho}} & if \; 0 < A_{t+1} < 1 \\ \left(\frac{(1-\alpha)a(E_{t+1})}{a'(E_{t+1})\left[\alpha (A_{t+1}H_{t+1})^{\rho} + (1-\alpha)P_{t+1}^{\rho}\right]}\right)^{\frac{1}{1-\rho}} & if \; A_{t+1} \geq 1 \end{cases}$$

The system of nine equations above is rather complex since we face with some implicit functions of infant surviving probability, human capital formation, growth rate of technomogical progress, environmental regeneration, and aggregate pollution. To study the global dynamics we need parametric forms, which is suggested in the next section, for these functions. And then a simulation should be carried out.

4. Suggestion for a numerical study

So far we have presented the theoretical model where some functions are defined implicitly. To be able to simulat the model we have to reveal the parametric functional forms for $h(e_{t+1}, g_{t+1})$, $g(e_t, L_t)$, $\pi(e_t, E_t)$, $\Omega(E_t)$, and $a(E_t)$. It is hard to come up with. We inherit develop the work by Lagerlof (2007) for the functional forms of $h(e_{t+1}, g_{t+1})$ and $g(e_t, L_t)$ which seem intuitive

$$h_{t+1} = \frac{\tau\phi^q + \phi^e e_{t+1}}{\tau\phi^q + \phi^e e_{t+1} + g_{t+1}} \tag{18}$$

where, as in Lagerlof (2007), $\tau \in (0.1)$ is exogenous and can be interpreted as a part of the fixed time cost, ϕ^q, that helps build human capital, so that $\tau\phi^q + \phi^e e_{t+1}$ can be thought of as effective education. Applying the expression defining optimal education in (9) to parametric form in (18), we can derive tha optimal education level, e_{t+1}, as

$$e(g_{t+1}, \pi_t) = max\left\{0,\ \frac{\sqrt{g_{t+1}(\frac{\phi^b}{\pi_t} + (1-\tau)\phi^q)} - \tau\phi^q}{\phi^e}\right\}$$

Next, let technological progress take the form

$$g_{t+1} = g(e_t, L_t) = (\tau\phi^q + \phi^e e_t)\xi(L_t)$$

where $\xi'(L_t) > 0$. The scale effect could take the functional form as

$$\xi(L_t) = min\{\theta L_t, \xi^*\}$$

where $\theta, \xi^* > 0$. This implies that the population affects technological progress linearly for $L_t \leq \xi^*/\theta$, and then stays flat.

Let the surviving probability of children at birth take the form

$$\pi_t = \pi(e_t, E_t) = \frac{e_t + E_t}{1 + e_t + E_t}$$

and the renewable function of environment

$$\Omega(E_t) = -E_t^2 + \beta E_t + \omega$$

where $\beta, \omega > 0$. And the total factor productivity takes the form

$$a(E_t) = E_t^\sigma,\ \sigma \in (0, 1)$$

Hence the dynamics of education for a fixed population is

$$e_{t+1} = e(g_{t+1}, \pi_t) = max\left\{0,\ \frac{\sqrt{(\tau\phi^q + \phi^e e_t)\xi(L_t)(\frac{\phi^b}{\pi_t} + (1-\tau)\phi^q)} - \tau\phi^q}{\phi^e}\right\} \equiv \varrho(e_t, L_t)$$

Now the system of global dynamics become

$$n_t = \frac{\gamma}{\frac{\phi^b}{\pi_t} + \phi^q + \phi^e e_{t+1}}$$

$$\pi_t = \pi(e_t, E_t) = \frac{e_t + E_t}{1 + e_t + E_t}$$

$$e_{t+1} = e(g_{t+1}, \pi_t) = max\left\{0, \ \frac{\sqrt{(\tau\phi^q + \phi^e e_t)\xi(L_t)(\frac{\phi^b}{\pi_t} + (1-\tau)\phi^q)} - \tau\phi^q}{\phi^e}\right\}$$

$$h_{t+1} = \frac{\tau\phi^q + \phi^e e_{t+1}}{\tau\phi^q + \phi^e e_{t+1} + g_{t+1}}$$

$$g_{t+1} = g(e_t, L_t) = (\tau\phi^q + \phi^e e_t)\xi(L_t)$$

$$A_{t+1} = (1 + g_{t+1})A_t$$

$$L_{t+1} = n_t L_t$$

$$E_{t+1} = E_t - E_t^2 + \beta E_t + \omega - P_{t+1}$$

$$P_{t+1} = \begin{cases} \left(\frac{(1-\alpha)A_{t+1}^\rho\left[(1+\beta)E_t - E_t^2 + \omega - P_{t+1}\right]}{\sigma\left[\alpha H_{t+1}^\rho + (1-\alpha)(A_{t+1}P_{t+1})^\rho\right]}\right)^{\frac{1}{1-\rho}} & if \ 0 < A_{t+1} < 1 \\ \left(\frac{(1-\alpha)\left[(1+\beta)E_t - E_t^2 + \omega - P_{t+1}\right]}{\sigma\left[\alpha(A_{t+1}H_{t+1})^\rho + (1-\alpha)P_{t+1}^\rho\right]}\right)^{\frac{1}{1-\rho}} & if \ A_{t+1} \geq 1 \end{cases}$$

5. Conclusion

This paper develops a unified endogenous growth theory to explain the evolutions of fertility, infant mortality, education, technology, pollution and growth of output along the thousands of years of the human history. The model interprets qualitatively well the take off from a Malthusian stagnation regime, to a demographic transition and then to a Modern Growth Regime in which the evolution of population is taken into account from evolutions of both fertility and infant mortality. Any economy starts the process of development with small population, very low level of technology and education will be trapped in a very low stable steady state characterized by very low growth rate of technological progress and zero-education. Because education is zero then the infant surviving probability is low even though environment quality might be good. Then the fertility rate is high due to the behavior of maximizing ultility of households. However, the population growth is still very low. Because the growth rate of technological progress is very low then there is no incentive for the households to invest in education for their children; and hence, along with small population, the growth rate of technological progress will be kept low in the next period and so on, and the population continues growing very slowly as well. This negative feedback loop locks the economy in very low and stable steady state for thousands of years. After thousands of years of development until the population size is moderate, the economy has possiblity to escape from the trap to converge to a high stable steady state characterized by high education, high growth rate of technological progress, low fertility rate and low infant mortality rate.

This paper treats pollution as an input for production of final output. During the period of demographic transition, there exists a turning point of pollution thank to accumulation of technology. When the level of technology is low, it associates with pollution to produce final output; and the technological progress makes the environment more polluted leading to lower the infant surviving probability and hence leading to the households giving more birth. But when level of technology exceeds a threshold, which is normalized by 1, the technology associates with aggregate human capital to produce final output; and in this case technological progress makes environment cleaner leading to an increase in infant surviving probability and hence leading to the households giving less birth. It is suggested from this paper that environmental quality plays a crucial role in demographic transition during the industrial revolution time in Western Europe.

The global dynamics of the system from theoretical model is quite complex and a more profound quantitative analysis is still a big challenge at this step. So this paper also suggests some functional forms for implicit functions in the theoretical model to be able to carry a numerical study in order to support the predictions from the theoretical model. Doing this simulation is still in my research agenda.

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Appendix

The Lagrangian for optimization problem of households

$$\mathcal{L} = \gamma\left[ln(n_t) + lnh(e_{t+1}, g_{t+1})\right] + (1-\gamma)ln(w_t h_t(1 - n_t[\frac{\phi^b}{\pi_t} + \phi^q + \phi^e e_{t+1}]))$$

$$+\lambda_1 w_t h_t(1 - n_t[\frac{\phi^b}{\pi_t} + \phi^q + \phi^e e_{t+1}]) + \lambda_2 n_t + \lambda_3 e_{t+1}$$

The Kuhn-Tucker conditions

$$\frac{\partial\mathcal{L}}{\partial n_t} = \frac{\gamma}{n_t} - \frac{(1-\gamma)[\frac{\phi^b}{\pi_t} + \phi^q + \phi^e e_{t+1}]}{1 - n_t[\frac{\phi^b}{\pi_t} + \phi^q + \phi^e e_{t+1}]} + \lambda_1 w_t h_t[\frac{\phi^b}{\pi_t} + \phi^q + \phi^e e_{t+1}] + \lambda_2 = 0 \quad (19)$$

$$\frac{\partial\mathcal{L}}{\partial e_{t+1}} = \frac{\gamma h_e(e_{t+1}, g_{t+1})}{h(e_{t+1}, g_{t+1})} - \frac{(1-\gamma)n_t\phi^e}{1 - n_t[\frac{\phi^b}{\pi_t} + \phi^q + \phi^e e_{t+1}]} - \lambda_1 w_t h_t n_t \phi^e + \lambda_3 = 0 \quad (20)$$

$$\lambda_1 w_t h_t(1 - n_t[\frac{\phi^b}{\pi_t} + \phi^q + \phi^e e_{t+1}]) = 0 \quad (21)$$

$$\lambda_2 n_t = 0 \quad (22)$$

$$\lambda_3 e_{t+1} = 0 \quad (23)$$

$$\lambda_1, \lambda_2, \lambda_3 \geq 0 \quad (24)$$

It is straightforward to show that $\lambda_2 = 0$ and $\lambda_1 > 0$. In effect, if $\lambda_2 > 0$ then $n_t = 0$, hence,

$$(1-\gamma)[\frac{\phi^b}{\pi_t}+\phi^q+\phi^e e_{t+1}]-\lambda_1 w_t h_t[\frac{\phi^b}{\pi_t}+\phi^q+\phi^e e_{t+1}]-\lambda_2=+\infty$$

which could not happen. Therefore, $\lambda_2 = 0$.

The two first-order conditions now are

$$\frac{\partial \mathcal{L}}{\partial n_t}=\frac{\gamma}{n_t}-\frac{(1-\gamma)[\frac{\phi^b}{\pi_t}+\phi^q+\phi^e e_{t+1}]}{1-n_t[\frac{\phi^b}{\pi_t}+\phi^q+\phi^e e_{t+1}]}+\lambda_1 w_t h_t[\frac{\phi^b}{\pi_t}+\phi^q+\phi^e e_{t+1}]=0 \tag{25}$$

$$\frac{\partial \mathcal{L}}{\partial e_{t+1}}=\frac{\gamma h_e(e_{t+1},g_{t+1})}{h(e_{t+1},g_{t+1})}-\frac{(1-\gamma)n_t\phi^e}{1-n_t[\frac{\phi^b}{\pi_t}+\phi^q+\phi^e e_{t+1}]}-\lambda_1 w_t h_t n_t \phi^e+\lambda_3=0 \tag{26}$$

If $\lambda_1 > 0$ then from (21) we have

$$n_t=\frac{1}{\frac{\phi^b}{\pi_t}+\phi^q+\phi^e e_{t+1}}$$

then (25) is equivalent to

$$\gamma[\frac{\phi^b}{\pi_t}+\phi^q+\phi^e e_{t+1}]+\lambda_1 w_t h_t[\frac{\phi^b}{\pi_t}+\phi^q+\phi^e e_{t+1}]=+\infty$$

which could not happen.

- If $\lambda_1 = \lambda_3 = 0$, then

$$n_t=\frac{\gamma}{\frac{\phi^b}{\pi_t}+\phi^q+\phi^e e_{t+1}}$$

$$h_e(e_{t+1},g_{t+1})[\frac{\phi^b}{\pi_t}+\phi^q+\phi^e e_{t+1}]-\phi^e h(e_{t+1},g_{t+1})=0$$

- If $\lambda_1 = 0$, $\lambda_3 > 0$, then

$$n_t=\frac{\gamma}{\frac{\phi^b}{\pi_t}+\phi^q+\phi^e e_{t+1}} \tag{27}$$

$$\frac{\gamma h_e(e_{t+1},g_{t+1})}{h(e_{t+1},g_{t+1})}-\frac{(1-\gamma)n_t\phi^e}{1-n_t[\frac{\phi^b}{\pi_t}+\phi^q+\phi^e e_{t+1}]}=-\lambda_3<0 \tag{28}$$

Equations (27) and (28) give us

$$h_e(e_{t+1},g_{t+1})[\frac{\phi^b}{\pi_t}+\phi^q+\phi^e e_{t+1}]-\phi^e h(e_{t+1},g_{t+1})<0$$