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Intertemporal equilibrium with financial asset and physical capital

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Abstract

We build an infinite-horizon dynamic deterministic general equilibrium model with imperfect market (borrowing constraint) in which heterogenous agents invest in capital or financial asset, work and consume, a representative firm maximizes profit. Firstly, the existence of intertemporal equilibrium is proved even if aggregate capital is not uniformly bounded. Secondly, we define and study financial bubble and physical bubble. Both financial and physical bubbles may occur but they can not occur at the same time. Lastly, we study impact of physical productivity and financial dividend on the physical market which we call *"the race between productivity and dividend"*.

Keywords: Financial asset bubble, capital asset bubble, intertemporal equilibrium, productivity, infinite horizon.

1 Introduction

2007-2012 recession requires us to reconsider some fundamental problems. What is an asset price bubble ? what is the root of bubble? Other fundametal problem is about the role of financial market on aggregate economic activity. Financial market is considered to be one of main causes of economic recession. But, does financial market always cause an economic recession? What is the role of financial market on productive sector?

Our approach to study these problems is to construct a dynamic deterministic general equilibrium with heterogenous agents which has financial asset, physical capital and endogenous labor supply. Consumers differ in discount factor, initial wealth[1]. Heterogeneous consumers invest, work and consume. They have two choices to invest: productive sector and financial sector. At date t, if one invests to physical capital then at date t+1, he (or she) will receive a return that depends on productivity of the economy. In the financial

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[1] A detailed survey on the effects of heterogeneity in macroeconomics can be found in [Guv12]

market, if one buys one unit of financial asset then at next date he (or she) will be able to resell this asset and also receive a dividend in term of consumption good. We introduce a borrowing constraint on financial asset under which, financial market is incomplete.

The first contribution of our paper concerns the existence of intertemporal equilibrium. Becker, Boyd III, and Foias ([BBIF91]) have demonstrated the existence of an intertemporal equilib-rium under borrowing constraints with inelastic labor supply. Kubler and Schmeider ([KS03]) have constructed and proved the existence of Markov equilibrium in infinite-horizon asset pricing model with incomplete market and collateral. Such Markov equilibrium is also proved to be competitive equilibrium. Becker, Bosi, Le Van and Seegmuller ([BBLVS11]) proved the existence of a Ramsey equilibrium with endogenous labor supply and borrowing constraint on physical asset. In these paper, they need an assumption (on endowment in [KS03], and on production function in [BBIF91], [BBLVS11]) under which aggregate capital and consumption are uniformly bounded.
Our framework is rich enough to cover all endogenous labor supply, physical capital, and financial asset in imperfect market which has borrowing constraint. To prove the existence of equilibrium, we firstly prove existence of equilibrium in each T - truncated economy. Then this sequence of equilibria has a limit for product topology, such limit is proved to be intertemporal equilibrium.
An other value added of our paper is that we don't need that aggregate capital and consumption are uniformly bounded to ensure the existence of intertemporal equilibrium.

Our second contribution is about asset bubbles. We focus on bubbles in rational expectation model with heterogenous agents. A survey on models to study bubble as asymmetric information, overlapping generation, heterogeneous-beliefs can be found in [BO12].
We say that there is a bubble on an asset if the price of this asset is greater its fundamental value. As we know, bubble does not exist neither in finite horizon rational expectation model nor in model without credit friction. In [BBLVS11], they define physical capital bubble in a Ramsey model. In their framework, under very mild conditions, physical capital bubble rules out. In rational expectation model without endowment, Tirole ([Tir82]) proved that there is no financial asset bubble. Le Van and Vailakis ([LVY12]) study financial asset bubble in an infinite horizon model which has endowment consumption and financial market with borrowing constraint. Diffirent from [Tir82], there is a room for financial asset bubble in [LVY12].

In our framework, we define and study both financial asset bubble and physical capital asset bubble. Both bubbles can occur but they can not occur at the same time. We prove that if there is physical asset bubble then in future, no one invests to productive sector. Hence, the economy will become financial market without productive sector. Consequently, there will be no financial asset bubble. We also give conditions on exogenous variables under which there is financial asset bubble at any equilibrium.

In our framework, the roots of asset bubbles are agents' heterogeneity, borrowing contraints with infinite horizon.

Analysing the relationships between financial market and productive sector is our third contribution. We study this problem by using a framework with exogenous labor supply
By an economic recession, we mean a situation in which aggregate capital is less than a given level, say K. There are many sources for economic recession as war, policy shocks, financial shocks..., but we focus on physical productivity. We show that if physical productivity at level K is lower than depreciation rate, i.e., $F'(K) \leq \delta$ and financial dividend is not so low

then economic recession will appear at infinitly many date.
Futhermore, *the race between financial dividend and physical productivity* is completely studied

- When $F'(0) < \delta$, and dividend of financial asset is not so small then capital market will disappear at infinitly many date.
- When $F'(0) < \dfrac{1}{\beta} - 1 + \delta$ we have an example in which dividend of financial asset is **high** and no investment in capital.
- When $F'(0) = \infty$ then the economy will invest at any period whatever are the dividends.
- Given $F'(0)$ finite, if ξ_t is very large w.r.t. ξ_{t+1} then $K_{t+1} > 0$.

Related literature:

Doblas-Madrid [DM12]: This paper presents a model of speculative bubbles where rational agents buy an overvalued asset becausegiven their private informationthey believe they have a good chance of reselling at a profit to a greater fool.

Martin and Ventura ([MV12]), Ventura [Ven12]: borrowing constraint entrepreneurs can borrow only a fraction of their future firm value. Once financing constraints are present, bubbles not only have a crowding out effect, but can also have a crowding-in effect, and thus allow a productive subset of entrepreneurs to increase investments.

Many significant researchs have explained why credit market frictions can make impact on aggregate econnomic activity ([GK10],[CMR10])

Gabaix [Gab11] proposes that idiosyncratic firm-level shocks can explain an important part of aggregate movements.

Basu, Pascali, Schiantarelli, Serven [BPSS12] shows that aggregate TFP, appropriately defined, and the capital stock can be used to construct sufficient statistics for the welfare of a representative consumer.

2 Model

The model is an infinite-horizon general equilibrium model without uncertainty. There are two types of agents: a representative firm without market power and m households.

2.1 Households

Each household invests in physical asset or financial asset, works, and consumes.

Consumption good: at each period t, price of consumption is denoted by p_t and agent i consumes $c_{i,t}$ units of consumption good.

Capital: at time t, if agent i decides to buy $k_{i,t+1} \geq 0$ units of new capital, then at period $t+1$, after having depreciated, agent i will receive $(1-\delta)k_{i,t+1}$ units of old capital, and $k_{i,t+1}$ units of old capital can be sold with price r_{t+1}.

Assumption 1. *At initial period 0, each household i has a initial capital endowments which is positive, i.e. $k_{i,0} \geq 0$ for $i = 1, \ldots, m$, and $\sum_{i=1}^{m} k_{i,0} > 0$.*

Financial assets: at period t if agent i decides to invest $a_{i,t} \geq 0$ units of financial asset with price q_t then she will receive ξ_{t+1} units of consumption good as dividend and she will able to sell $a_{i,t}$ units of financial asset with price q_{t+1}. These assets may be lands, houses...

Assumption 2. *For every $t \geq 0, 0 < \xi_t < \infty$.*

Assumption 3. *At the initial period 0, each household i has a initial financial asset endowments who is positive, i.e. $a_{i,-1} \geq 0$ for $i = 1, \ldots, m$, and $\sum_{i=1}^{m} a_{i,-1} = 1$.*

In [BBLVS11], they required $k_{i,0} > 0$ for every i. By constrat, we don't need $k_{i,0} > 0$.

Labor: at each period t, each household i is endowed 1 unit of leisure-time, leisure demand of agent i is denoted by $\lambda_{i,t}$ and the individual labor supply is given by $l_{i,t} = 1 - \lambda_{i,t}$. The price of labor is denoted by w_t at each period t.

Table 1: Household i's balance sheet at date t

	Expenditures	Revenues	
Consumption	$p_t c_{i,t}$	$w_t(1-\lambda_{i,t})$	wage
Capital investment	$p_t(k_{i,t+1} - (1-\delta)k_{i,t})$	$r_t k_{i,t}$	capital return from date $t-1$
Financial asset	$q_t a_{i,t}$	$(q_t + p_t\xi_t)a_{i,t-1}$	financial delivery from date $t-1$

Each household i takes sequences of prices $(p, r, w, q) = (p_t, r_t, w_t, q_t)_{t=0}^{\infty}$ as given and solves

$$(P_i(p,r,w,q)): \quad \max_{\left((c_{i,t},k_{i,t},\lambda_{i,t},a_{i,t})_{i=1}^{m}\right)_{t=0}^{+\infty}} \left[\sum_{t=0}^{+\infty} \beta_i^t u_i(c_{i,t}, \lambda_{i,t})\right] \tag{1}$$

$$\text{subject to} \quad k_{i,t+1} \geq 0, a_{i,t} \geq 0, \lambda_{i,t} \in [0,1], \tag{2}$$

$$\text{(budget constraints)} \quad p_t(c_{i,t} + k_{i,t+1} - (1-\delta)k_{i,t}) + q_t a_{i,t} \leq r_t k_{i,t} + (q_t + p_t\xi_t)a_{i,t-1} + w_t(1-\lambda_{i,t}), \tag{3}$$

where β_i is the discount factor of agent i and u_i such that

Assumption 4. *$u_i : \mathbb{R}_+^2 \to \mathbb{R}$ is C^0, strictly increasing and concave.*

2.2 Firms

We consider a representative firm with no market power. The technology is represented by a constant returns to scale production function: $F(K_t, L_t)$, where K_t, L_t are the aggregate

capital and the aggregate labor.
For each period, firm takes prices (p_t, r_t, w_t) as given and solves

$$(P(r_t, w_t)) : \qquad \max_{K_t \geq 0, L_t \geq 0} \Big[p_t F(K_t, L_t) - r_t K_t - w_t L_t\Big] \tag{4}$$

Assumption 5. *1. $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ is C^1, constant returns to scale, strictly increasing and strictly concave.*

2. Inputs are essential: $F(0, 0) = 0$.

3. $F(K, L) \to +\infty$ when $L > 0$ and $K \to +\infty$ or when $K > 0$ and $L \to +\infty$.

2.3 Equilibrium

We define an infinite-horizon sequences of prices and quantities is

$$(p, r, w, q, (c_i, k_i, \lambda_i, a_i)_{i=1}^m, K, L)$$

where for each $i = 1, \ldots, m$

$$\begin{aligned}
(c_i, k_i, \lambda_i, a_i) &:= \Big((c_{i,t})_{t=0}^{+\infty}, (k_{i,t})_{t=0}^{+\infty}, (w_{i,t})_{t=0}^{+\infty}, (a_{i,t})_{t=0}^{+\infty}\Big) \in \mathbb{R}_+^{+\infty} \times \mathbb{R}^{+\infty} \times \mathbb{R}_+^{+\infty} \times \mathbb{R}^{+\infty},\\
(p, r, w, q) &:= \Big((p_t)_{t=0}^{+\infty}, (r_t)_{t=0}^{+\infty}, (w_t)_{t=0}^{+\infty}, (q_t)_{t=0}^{+\infty}\Big) \in \mathbb{R}^{+\infty} \times \mathbb{R}_+^{+\infty} \times \mathbb{R}_+^{+\infty} \times \mathbb{R}^{+\infty},\\
(K, L) &:= \Big((K_t)_{t=0}^{+\infty}, (L_t)_{t=0}^{+\infty}\Big) \in \mathbb{R}_+^{+\infty} \times \mathbb{R}_+^{+\infty}.
\end{aligned}$$

We also denote $z_0 := (p, r, w, q), z_i := (c_i, k_i, \lambda_i, a_i)$ for each $i = 1, \ldots, m$, $z_{m+1} = (K, L)$ and $z = (z_i)_{i=0}^{m+1}$.

Definition 1. *A sequence of prices and quantities $\Big(\bar{p}_t, \bar{r}_t, \bar{w}_t, \bar{q}_t, (\bar{c}_{i,t}, \bar{k}_{i,t}, \bar{\lambda}_{i,t}, \bar{a}_{i,t})_{i=1}^m, \bar{K}_t, \bar{L}_t\Big)_{t=0}^{+\infty}$ is an equilibrium of the economy $\mathcal{E} = \Big((u_i, \beta_i, k_{i,0}, a_{i,-1})_{i=1}^m, F,\Big)$ if*

(i) Price positivity: $\bar{p}_t, \bar{r}_t, \bar{w}_t, \bar{q}_t > 0$ for $t \geq 0$.

(ii) Market clearing: at each $t \geq 0$,

$$\textit{good}: \quad \sum_{i=1}^m (\bar{c}_{i,t} + \bar{k}_{i,t+1} - (1-\delta)\bar{k}_{i,t}) = F(\bar{K}_t, \bar{L}_t) + \xi_t, \tag{5}$$

$$\textit{capital}: \quad \bar{K}_t = \sum_{i=1}^m \bar{k}_{i,t}, \tag{6}$$

$$\textit{labor}: \quad \bar{L}_t = \sum_{i=1}^m \bar{l}_{i,t}, \tag{7}$$

$$\textit{financial asset}: \quad 1 = \sum_{i=1}^m \bar{a}_{i,t}. \tag{8}$$

(iii) Optimal consumption plans: for each i, $\Big((\bar{c}_{i,t}, \bar{k}_{i,t}, \bar{\lambda}_{i,t}, \bar{a}_{i,t})_{i=1}^m\Big)_{t=0}^{\infty}$ is a solution of the problem $(P_i(\bar{p}, \bar{r}, \bar{w}, \bar{q}))$.

(iv) Optimal production plan: for each $t \geq 0$, $(\bar{K}_t, \bar{L}_t)$ is a solution of the problem $(P(\bar{r}_t, \bar{w}_t))$.

Lemma 2.1. *Labor supply is bounded.*

Proof. It is clear since $l_{i,t} = 1 - \lambda_{i,t} \in [0,1]$ and $\sum_{i=1}^{m} l_{i,t} \in [0, m]$. □

The following result proves that aggregate capital and consumption are bounded for the product topology.

Lemma 2.2. *Individual and aggregate capital supplies and consumptions are in a compact set for the product topology. Moreover, they are uniformly bounded if* $\frac{\partial F}{\partial K}(\infty, m) < \delta$ *and* $(\xi_t)_t$ *are uniformly bounded.*

Proof. Denote

$$\begin{aligned} D_0(F,\delta,K_0,\xi_0) &:= F(K_0,m) + (1-\delta)K_0 + \xi_0, \quad (9)\\ D_t(F,\delta,K_0,\xi_0,\dots,\xi_t) &:= F(D_{t-1}(F,\delta,K_0,\xi_0,\dots,\xi_{t-1}),m)\\ &\quad +(1-\delta)D_{t-1}(F,\delta,K_0,\xi_0,\dots,\xi_{t-1}) + \xi_t \quad \forall t \geq 0. \quad (10)\end{aligned}$$

Then $\sum_{i=1}^{m} c_{i,t} + K_{t+1} \leq D_t$ for every $t \geq 0$.

We now consider the case where $\frac{\partial F}{\partial K}(\infty, m) < \delta$ and there exists $\xi > 0$ such that $\xi_t \leq \xi$. We are going to prove that $0 \leq K_t \leq \max\{K_0, x\} =: K$ where x such that[2] $F(x,m) + (1-\delta)x + \xi = x$. Note that if $y \geq x$ then $F(y,m) + (1-\delta)y + \xi \leq y$.
We have

$$\begin{aligned} K_{t+1} = \sum_{i=1}^{m} k_{i,t+1} &\leq F(K_t,m) + (1-\delta)K_t + \xi_t\\ &\leq F(K_t,m) + (1-\delta)K_t + \xi.\end{aligned}$$

Then $K_1 \leq F(K_0,m) + (1-\delta)K_0 + \xi \leq F(K) + (1-\delta)K + \xi \leq K$. Iterating the argument, we find $K_t \leq K$ for each $t \geq 0$.
Consumptions are bounded because $\sum_{i=1}^{m} c_{i,t} \leq F(K_t,m) + (1-\delta)K_t + \xi$. □

3 The existence of equilibrium

We follow Becker, Bosi, Le Van et Seegmuller ([BBLVS11]). First, we prove the existence of equilibrium for each $T-$ truncated economy $\mathcal{E}^T$. Second, we show that this sequence of equilibriums converges for the product topology to an equilibrium of our economy $\mathcal{E}$. The value-added in our proof is that we do not need aggregate consumption and capital are uniformly bounded.

In order to prove the existence of equilibrium for $T-$ truncated economy $\mathcal{E}^T$, we prove the existence of the bounded economy $\mathcal{E}_b^T$ and then by using the concavity of the utility function, we will prove that such equilibrium is also an equilibrium of $\mathcal{E}^T$.

[2] since $F(\cdot, m)$ is concave, x is unique. The existence of x is ensured by $F(0,m)+\xi > 0$ and $\lim_{x\to\infty} F(x,m) - \delta x + \xi < 0$

3.1 The existence of equilibrium for $T-$ truncated economy $\mathcal{E}^T$

We define $T-$ truncated economy $\mathcal{E}^T$ as $\mathcal{E}$ but there are no activities from period $T+1$ to the infinity, i.e., $c_{i,t} = b_{i,t-1} = k_{i,t} = \lambda_{i,t} = K_t = 0$ for every $i = 1, \ldots, m$, $t \geq T+1$.
Then we define the bounded economy $\mathcal{E}_b^T$ as $\mathcal{E}^T$ but all variables (consumption demand, capital supply, leisure demand, asset investment, capital and labor demands) are bounded in the following bounded sets:

$$\begin{aligned}
\mathcal{C}_i &:= [0, B_c]^{T+1}, \quad B_c > m + F(B_K) + (1-\delta)B_K + \max_{t \leq T} \xi_t, \\
\mathcal{K}_i &:= [0, B_k]^T, \quad B_k > D^T := \max_{t \leq T} D_t(K_0, \xi_0, \ldots, \xi_t) \\
\Lambda_i &:= [0, 1]^{T+1}, \\
\mathcal{A}_i &:= [0, B_a]^T, \quad B_a > 1 \\
\mathcal{K} &:= [0, B_K]^{T+1}, \quad B_K > mB_k \\
\mathcal{L} &:= [0, B_L]^{T+1}, \quad B_L > m.
\end{aligned}$$

Proposition 3.1. *Under Assumption 1, 2, 3, 4, 5, there exists an equilibrium for $\mathcal{E}_b^T$.*

Proof. See Appendix A. We adapt the method of proof given by Florenzano (1999) and, in particular, we use the Gale - Mas Colell lemma. □

Theorem 3.1. *An equilibrium of $\mathcal{E}_b^T$ is an equilibrium for $\mathcal{E}^T$.*

Proof. We follow Becker, Bosi, Le Van, et Seegmuller ([BBLVS11]).
Let $\left(\bar{p}_t, \bar{r}_t, \bar{w}_t, \bar{q}_t, (\bar{c}_{i,t}, \bar{k}_{i,t}, \bar{\lambda}_{i,t}, \bar{a}_{i,t})_{i=1}^m, \bar{K}_t, \bar{L}_t\right)_{t=0}^T$ is an equilibrium of $\mathcal{E}_b^T$. Note that $k_{i,T+1} = a_{i,T} = 0$ for every $i = 1, \ldots, m$. We can see that conditions (i) and (ii) in Definition 1 are hold. We will show that conditions (iii) and (iv) in Definition 1 are hold.
For Condition (iii), let $z_i = \left((c_{i,t}, k_{i,t}, \lambda_{i,t}, a_{i,t})_{i=1}^m\right)_{t=0}^T$ is a plan of household i such that budget constraints for each $t = 0, \ldots, T$.
Assume that $\sum\limits_{t=0}^T \beta_i^t u_i(c_{it}, \lambda_{i,t}) > \sum\limits_{t=0}^T \beta_i^t u_i(\bar{c}_{it}, \bar{\lambda}_{i,t})$. For each $\gamma \in (0,1)$, we define $z_i(\gamma) := \gamma z_i + (1-\gamma)\bar{z}_i$. By definition of $\mathcal{E}_b^T$, we can choose γ sufficiently close to 0 such that $z_i(\gamma) \in \mathcal{C}_i \times \mathcal{K}_i \times \Lambda_i \times \mathcal{A}_i$. It is clear that $z_i(\gamma)$ is satisfied budget constraints.
By the concavity of the utility function, we have

$$\begin{aligned}
\sum_{t=0}^T \beta_i^t u_i(c_{it}(\gamma), \lambda_{i,t}(\gamma)) &\geq \gamma \sum_{t=0}^T \beta_i^t u_i(c_{it}, \lambda_{i,t}) + (1-\gamma) \sum_{t=0}^T \beta_i^t u_i(\bar{c}_{it}, \bar{\lambda}_{i,t}) \\
&> \sum_{t=0}^T \beta_i^t u_i(\bar{c}_{it}, \bar{\lambda}_{i,t}).
\end{aligned}$$

Contradiction to the optimality of $\bar{v}_i$. So, we have shown that conditions (iii) in Definition 1 is hold. A similarly proof of conditions (iv) in Definition 1 permits us finish our proof of Theorem. □

3.2 The existence of equilibrium in $\mathcal{E}$

To take the limit of sequence of equilibria, we need the following assumption

Assumption 6. *The utility function* u_i *take the separable form* $u_i(c_{i,t})+v_i(\lambda_{i,t})$ *with* $u_i, v_i : \mathbb{R}_+ \to \mathbb{R}$*, and* $u_i, v_i \in \mathbb{C}^1$*. In addition, we assume* $u_i(0) = v_i(0) = 0$*,* $u_i'(0) = v_i'(0) = +\infty$*,* $u_i'(c_{i,t}), v_i'(\lambda_{i,t}) > 0$ *for* $c_{i,t}, \lambda_{i,t}$*. Finally,* u_i, v_i *are concave.*

Assumption 7. *For each* i*, utility of agent* i *is finite*

$$\sum_{t=0}^{\infty} \beta_i^t u_i(D_t(F, \delta, K_0, \xi_0, \dots, \xi_t)) < \infty. \tag{11}$$

Remark 3.1. *Assumption 7 is hold if there exists* $b < \infty$ *such that, for every* $i \in \{1, \dots, m\}$

$$\sum_{t=0}^{\infty} \beta_i^t \max_{s \le t}\{\xi_s\} < \infty, \tag{12}$$

$$\sum_{t=0}^{\infty} \beta_i^t (F'(b, m) + 1 - \delta)^t \max_{s \le t}\{\xi_s, 1\} < \infty. \tag{13}$$

Proof. Indeed, assume that there exists $b < \infty$ such that (12) and (13) for every i. Denote $A = F'(b, m), B = F(b, m)$. By using $F(\cdot, m)$ is increasing and concave, we obtain $F(x, m) \le Ax + B$ for every $x \ge 0$. Since definition of $D_t(K_0, \xi_0, \dots, \xi_t)$, we have

$$\begin{aligned} D_t(F, \delta, K_0, \xi_0, \dots, \xi_t) &\le (A + 1 - \delta)^{t+1} K_0 + (A + 1 - \delta)^t (B + \xi_0) + \dots + (B + \xi_t) \\ &\le (A + 1 - \delta)^{t+1} K_0 + (B + \max_{s \le t}\{\xi_s\}) \sum_{s=0}^{t} (A + 1 - \delta)^s \end{aligned}$$

Since u_i is concave, there exists $a_i > 0, b_i > 0$ such that $u_i(x) \le a_i x + b_i$ for every $x \ge 0$. Then

$$\sum_{t=0}^{\infty} \beta_i^t u_i(D_t(F, \delta, K_0, \xi_0, \dots, \xi_t)) \le \sum_{t=0}^{\infty} \beta_i^t \Big(a_i D_t(F, \delta, K_0, \xi_0, \dots, \xi_t) + b_i). \tag{14}$$

Case 1: $A \le \delta$ then $D_t(F, \delta, K_0, \xi_0, \dots, \xi_t) \le K_0 + (t + 1)(B + \max_{s \le t} \xi_s)$. Combining with (12), (13), and (14) we obtain (11).
Case 2: $A > \delta$, then

$$\begin{aligned} D_t(F, \delta, K_0, \xi_0, \dots, \xi_t) &= (A + 1 - \delta)^{t+1} K_0 + (A + 1 - \delta)^t (B + \xi_0) + \dots + (B + \xi_t) \\ &\le (A + 1 - \delta)^{t+1} K_0 + (B + \max_{s \le t}\{\xi_s\}) \frac{(A + 1 - \delta)^{t+1} - 1}{A - \delta}. \end{aligned}$$

Combining with (12), (13), and (14) we obtain (11). □

Note that there exists some cases with $F'(\infty, m) > \delta$ and $(\xi_t)_t$ are not uniformly bounded, conditions (12) and (13) are still hold. For example, if there exists $b < \infty$ such that $\beta_t < \beta_i(F'(b, m)+1-\delta) < 1$ and $\xi_t \le \alpha^t$ where $\alpha > 1$ such that $\alpha\beta_i(F'(b, m)+1-\delta) < 1$ then conditions (12) and (13) are hold.

Theorem 3.2. *Under Assumption 1, 2, 3, 5, 6 and 7, there exists an equilibrium in the infinite-horizon economy with endogenous labor supply, borrowing constraints and zero debt constraints.*

Proof. See Appendix B. We consider the limit of sequences of equilibria in $\mathcal{E}^T$, when $T \to \infty$. We use convergence for the product topology. □

3.3 Model with exogenous labor supply

In this section, we consider a model in which labor is exogenous in order to study the impact of productivity and financial asset on the capital market.

Each household i takes sequences of prices $(p, r, q) = (p_t, r_t, q_t)_{t=0}^{\infty}$ as given and solves the following problem

$$\begin{aligned}
(P_i(p,r,q)): \quad & \max_{\left((c_{i,t},k_{i,t},a_{i,t})_{i=1}^m\right)_{t=0}^{+\infty}} \left[\sum_{t=0}^{+\infty} \beta_i^t u_i(c_{i,t})\right] && (15)\\
\text{subject to} \quad & k_{i,t+1} \geq 0, a_{i,t} \geq 0, && (16)\\
\text{(budget constraints)} \quad & p_t(c_{i,t} + k_{i,t+1} - (1-\delta)k_{i,t}) + q_t a_{i,t} \\
& \leq r_t k_{i,t} + (q_t + p_t \xi_t) a_{i,t-1} + \theta^i \pi_t. && (17)
\end{aligned}$$

For each period, firm takes prices (p_t, r_t) as given and solves

$$(P(p_t, r_t)): \quad \max_{K_t \geq 0} \left[p_t F(K_t) - r_t K_t\right] \tag{18}$$

$(\theta^i)_{i=1}^m$ is the share of profit, $\theta^i \geq 0$ for all i and $\sum_{i=1}^m \theta^i = 1$.

In order to prove the existence of equilibrium, we only need the following assumptions (note that we don't need the assumption $\lim_{K \to 0} F'(k) = +\infty$)). The proof of existence of equilibrium is similar to the case in which labor is endogenous.

Assumption 8. *$F(\cdot)$ is strictly increasing, concave, $F(0) = 0$ and $F'(\infty) < \delta$.*

Assumption 9. *Initial endowment of each consumer and aggregate capital are strictly positive*

$$\begin{aligned}
1_{\{a_{i,-1}>0\}} + 1_{\{\theta^i>0\}} &> 0 \quad \forall i, && (19)\\
K_0 &> 0. && (20)
\end{aligned}$$

Since at equilibrium $p_t > 0$ for every $t \geq 0$, we can normalize by fixing $p_t = 1$ for every $t \geq 0$.

4 Bubbles

We give some general results about bubbles in this section. Other results will be presented in a model with exogenous labor.

4.1 Bubble on financial asset

We consider an equilibrium $(\bar{p}, \bar{r}, \bar{w}, \bar{q}, (\bar{c}_i, \bar{k}_i, \bar{\lambda}_i, \bar{a}_i)_{i=1}^m, \bar{K}, \bar{L})$. We observe that the Slater condition is satisfied, hence we can write the Kuhn-Tucker necessary conditions to all variables. Let $\bar{\mu}_{i,t}$ denote the multiplier associated with budget constraints of agent i at period t, and the multiplier of variable x is denoted by $\lambda(x)$. Note that $\bar{\mu}_{i,t} > 0$ for all $i = 1, \ldots, m$, and $t \geq 0$.

Since for each $t \geq 0$, $\sum\limits_{i=1}^{m} \bar{a}_{i,t} = 1$, hence for each $t \geq 0$, there there exists i_t such that $\bar{a}_{i(t),t} > 0$. Therefore $\lambda(\bar{a}_{i(t),t}) = 0$. FOCs of $\bar{a}_{i(t),t}$ and $\bar{c}_{i,t}$,$\bar{c}_{i,t+1}$ implies that

$$\frac{\bar{q}_t}{\bar{q}_{t+1} + \bar{p}_{t+1}\xi_{t+1}} = \frac{\bar{\mu}_{i(t),t+1}}{\bar{\mu}_{i(t),t}} = \max_{i\in\{1,\ldots,m\}} \frac{\bar{\mu}_{i,t+1}}{\bar{\mu}_{i,t}} = \gamma_t \frac{\bar{p}_t}{\bar{p}_{t+1}}, \tag{21}$$

where $\gamma_t := \max\limits_{i\in\{1,\ldots,m\}} \dfrac{\beta_i u_i'(\bar{c}_{i,t+1})}{u_i'(\bar{c}_{i,t})}$. Therefore, for each $t \geq 0$ we have

$$\frac{\bar{q}_t}{\bar{p}_t} = \gamma_t(\frac{\bar{q}_t}{\bar{p}_t} + \xi_{t+1}).$$

Consequently, for each $t \geq 1$

$$\frac{\bar{q}_0}{\bar{p}_0} = Q_t\frac{\bar{q}_t}{\bar{p}_t} + \sum_{s=1}^{t} Q_s\xi_s, \tag{22}$$

where $Q_t := \prod\limits_{s=0}^{t-1} \gamma_s$ is discount factor of the economy from initial period to period t, for each $t \geq 1$, and $Q_0 := 1$ by convention. $Q_t\dfrac{\bar{q}_t}{\bar{p}_t}$ is discounted price of financial asset in term of consumption good at period t. $\sum\limits_{s=1}^{t} Q_s\xi_s$ is present value of financial asset at period t. The sequence $(\sum\limits_{s=1}^{t} Q_s\xi_s)_{t=0}^{+\infty}$ is inscreasing and bounded by $\dfrac{\bar{q}_0}{\bar{p}_0}$, so there exists $\sum\limits_{t=1}^{+\infty} Q_t\xi_t$. On the other hand, (22) implies that $(Q_t\alpha_t)_{t=0}^{+\infty}$ is a descreasing and bounded sequence, hence there exists $\lim\limits_{t\to+\infty} Q_t\alpha_t$.

Definition 2. *The fundamental value* [3] *of financial asset*

$$FA_0 := \sum_{t=1}^{+\infty} Q_t\xi_t \tag{23}$$

Definition 3. *We say there is a bubble on financial asset if the price of financial asset is greater than its fundamental value:* $\dfrac{\bar{q}_0}{\bar{p}_0} > FA_0$, *i.e.,* $\lim\limits_{t\to+\infty} Q_t\alpha_t > 0$.

Lemma 4.1. $\lim\limits_{t\to+\infty} Q_t\dfrac{\bar{q}_t}{\bar{p}_t} > 0$ *if and only if* $\sum\limits_{t=1}^{+\infty} \dfrac{\bar{p}_t\xi_t}{\bar{q}_t} < +\infty$.

Proof. Denote $H_t := Q_t\dfrac{\bar{q}_t}{\bar{p}_t}$.

From definition of Q_t, we have $Q_t = Q_{t-1}\gamma_{t-1} = Q_{t-1}\dfrac{\bar{q}_{t-1}}{\bar{q}_{t-1}}\dfrac{\bar{p}_t}{\bar{q}_t + \bar{p}_t\xi_t}$. Hence, $H_{t-1} = \big(1 + \dfrac{\bar{p}_t\xi_t}{\bar{q}_t}\big)H_t$. Therefore,

$$H_0 = H_t\prod_{s=1}^{t}\big(1 + \frac{\bar{p}_s\xi_s}{\bar{q}_s}\big). \tag{24}$$

[3] By the same way, we have: for each $T > t$, $\dfrac{\bar{q}_t}{\bar{p}_t} = \alpha_t = Q_T^t\alpha_T + \sum\limits_{s=t+1}^{T} Q_s^t\xi_s$, where $Q_s^t := \prod\limits_{h=t}^{s-1}\gamma_h$ for each $s \geq t+1$, and $Q_t^t = 1$ by convention. And we can define the fundamental value of financial asset at date t by $FA_t := \sum\limits_{s=t}^{+\infty} Q_s^t\xi_s$. However, without loss of generality, we only consider at initial date

Since $H_0 = \bar{q}_0/\bar{p}_0 > 0$, we get that

$$\lim_{t\to+\infty} Q_t \frac{\bar{q}_t}{\bar{p}_t} > 0 \quad \text{if and only if} \quad \lim_{t\to+\infty} \prod_{s=1}^{t} \big(1 + \frac{\bar{p}_s \xi_s}{\bar{q}_s}\big) < +\infty, \tag{25}$$

$$\text{if and only if} \quad \sum_{t=1}^{+\infty} \frac{\bar{p}_t \xi_t}{\bar{q}_t} < +\infty. \tag{26}$$

□

Proposition 4.1. *If there exists $l \in \{1, \dots, m\}$ such that $\liminf_{t\to+\infty} \bar{a}_{l,t} > 0$ then there is no bubble on financial asset, i.e., $\lim_{t\to+\infty} Q_t \frac{\bar{q}_t}{\bar{p}_t} = 0$.*

Proof. Assume that there exists $l \in \{1, \dots, m\}$ such that $\liminf_{t\to+\infty} \bar{a}_{l,t} > 0$. Then there exists t_0 such that $\bar{a}_{l,t} > 0$ for every $t \geq t_0$.
If bubble occurs, then $\eta := \lim_{t\to+\infty} Q_t \frac{\bar{q}_t}{\bar{p}_t} > 0$. Therefore, there exists $t_1 \geq t_0$, and $\epsilon > 0$ such that $\bar{a}_{l,t} - \epsilon \frac{\eta}{p_t} Q_t q_t > 0$ for every $t \geq t_1$. We construct a new strategy for household l by keeping other choices except $\bar{c}_{l,t_1}$, and $(\bar{a}_{l,t})_{t\geq t_1}$

$$\begin{aligned} a_{l,t} &:= \bar{a}_{l,t} - \epsilon \frac{\eta p_t}{Q_t q_t} > 0, \quad t > t_1 \\ c_{l,t_1} &:= \bar{c}_{l,t_1} + \epsilon \frac{\eta p_t}{Q_t q_t}. \end{aligned}$$

Then this new strategy violates the optimality of $(\bar{c}_l, \bar{k}_l, \bar{a}_l, \bar{\lambda}_l)$, contradiction! □

We now give a necessary condition which depends on exogenous variables to the existence of financial bubble. For simplicity reasons, we write D_t instead of $D_t(F, \delta, K_0, \xi_0, \dots, \xi_t)$.

Theorem 4.1. (A Necessary Condition for Financial Bubble). *If there exists A such that $D_t \leq A\xi_t$ for every $t \geq 0$ then at any equilibrium there is no bubble on financial asset.*

Proof. Consider at an equilibrium. Without loss of generality, we can normalize by setting $p_t = 1$ for every $t \geq 0$. Assume that there exists A such that $D_t \leq A\xi_t$ for every $t \geq 0$, then we have

$$\sum_{t\geq 0}^{\infty} Q_t D_t \leq A \sum_{t=0}^{\infty} Q_t \xi_t < \infty.$$

On the other hand, we have $w_t(1 - \lambda_{i,t}) \leq w_t L_t \leq F(K_t, m) \leq D_t$ and $\sum_{i=1}^{m} c_{i,t} + K_{t+1} \leq F(K_t, L_t) + (1-\delta)K_t + \xi_t \leq D_t$, hence $\lim_{t\to\infty} K_{t+1} Q_t = 0$ and

$$\begin{aligned} \sum_{t\geq 0}^{\infty} w_t(1 - \lambda_{i,t}) Q_t &< \infty, \\ \sum_{t\geq 0}^{\infty} c_{i,t} Q_t &< \infty. \end{aligned}$$

Since budget constraint, we get for every $i = 1, \dots, m$ and $t \geq 0$

$$c_{i,t}Q_t + k_{i,t+1}Q_t + q_t a_{i,t}Q_t = (r_t + 1 - \delta)k_{i,t}Q_t + (q_t + \xi_t)a_{i,t-1}Q_t + w_t(1 - \lambda_{i,t})Q_t.$$

Let t run from 0 to T, then take the sum and noting that $k_{i,t+1}\Big[Q_t - (r_{t+1}+1-\delta)Q_{t+1}\Big] = 0$ and $q_tQ_t = (q_{t+1} + \xi_{t+1})Q_{t+1}$, we obtain

$$\begin{aligned}\sum_{t=0}^{T} c_{i,t}Q_t + k_{i,T+1}Q_t + a_{i,T}q_TQ_T &= (r_0 + 1 - \delta)k_{i,0} + (q_0 + \xi_0)a_{i,-1} \\ &\quad + \sum_{t=0}^{T} w_t(1 - \lambda_{i,t})Q_t. \end{aligned} \tag{27}$$

Consequently, there exists $\lim_{T\to\infty} a_{i,T}q_TQ_T$ for each $i = 1, \dots, m$.
If $\lim_{T\to\infty} q_TQ_T > 0$ then there exists $\lim_{T\to\infty} a_{i,T}$ for each $i = 1, \dots, m$. On the other hand, $\sum_{i=1}^{m} a_{i,t} = 1$ for every $t \geq 0$, hence there exists i such that $\lim_{T\to\infty} a_{i,T} > 0$. According Proposition 4.1, we have $\lim_{T\to\infty} q_TQ_T = 0$, contradiction! □

Corollary 1. *Assume that $F'(\infty, m) < \delta$ and that there exists $\xi, \bar{\xi} > 0$ such that $\xi \leq \xi_t \leq \bar{\xi}$ for every $t \geq 0$. Then at equilibrium, $\lim_{t\to\infty} Q_t\frac{q_t}{p_t} = 0$, i.e., there is no bubble on financial asset.*

Proof. Assume that $F'(\infty, m) < \delta$ and that there exists $\xi, \bar{\xi} > 0$ such that $\xi \leq \xi_t \leq \bar{\xi}$ for every $t \geq 0$. Then $D_t \leq K \leq \frac{K}{\xi}\xi_t$ for every $t \geq 0$, where K is given in the proof of Lemma 2.2. Using Theorem 4.1, we obtain the result. □

We now use the model with exogenous labor supply to study financial asset bubble. Before state a sufficient condition for financial bubble, we need the following technique result

Lemma 4.2. *The following function is decreasing on $(0, D_0)$*

$$f_i(x) := u_i'(x) - \beta_i\Big(F'(D_0 - x) + 1 - \delta\Big)u_i'\Big(F(D_0 - x) + (1 - \delta)(D_0 - x) + \xi_1\Big).$$

Moreover, the equation $f(x) = 0$ has a unique solution $x_i < D_0$ if

$$u_i'(D_0) - \beta_i\Big(F'(0) + 1 - \delta\Big)u_i'(\xi_1) < 0.$$

Proof. See Appendix C. □

The following Theorem suggests that when ξ_0 is very large with respect to K_0 and $(\xi_t)_{t\geq 1}$ then financial asset bubble occurs. Note that Le Van and Vailakis [LVY12] have given an bubble example in which $\sum_{t=1}^{\infty} \xi_t < \xi_0$. We theorically prove that under some conditions on exogenous variables, there is a bubble in financial asset.

Theorem 4.2. (A Sufficient Condition for Financial Bubble). *Consider model with exogenous labor supply. Assume that the two following conditions are true* *(i) For every $i \in \{1, \dots, m\}$,*

$$u_i'(D_0) - \beta_i\Big(F'(0) + 1 - \delta\Big)u_i'(\xi_1) < 0 \tag{28}$$

(ii) there exists $i \in \{1, \dots, m\}$ such that

$$\begin{aligned} & (r_0 + 1 - \delta)k_{i,0} + \theta^i \pi(K_0) + \xi_0 a_{i,-1} \\ \geq \; & \max(B_{i,1}, B_{i,2}) + (1 - a_{i,-1})q(F, \delta, K_0, (\xi_t)_{t\geq 0}), \end{aligned} \tag{29}$$

where

$$\begin{aligned} q(F, \delta, K_0, (\xi_t)_{t\geq 0}) \;&:=\; \sum_{t\geq 1} \frac{\xi_t}{\prod_{s=0}^{t-1}(F'(D_s) + 1 - \delta)} \\ B_{i,1} \;&:=\; (u_i')^{-1}\Big(\beta_i(F'(0) + 1 - \delta)u_i'(\xi_1)\Big) \\ B_{i,2} \;&:=\; x_i + (F')^{-1}\Big(\delta - 1 + \frac{\xi_1}{q(F, \delta, K_0, (\xi_t)_{t\geq 0})}\Big), \end{aligned}$$

where x_i is defined in Lemma 4.2,

then there is bubble in financial asset at any equilibirum.

Note that $q(F, \delta, K_0, (\xi_t)_{t\geq 0}) \leq \sum_{t\geq 1} \frac{\xi_t}{(F'(\infty) + 1 - \delta)^t}$ which does not depend on ξ_0.

Proof. Consider an equlibirum. Assume that there is no bubble in financial asset then we have $q_0 = \sum_{t\geq 1} Q_t \xi_t$.
For each $t \geq 0$, we have

$$\gamma_t = \max_i \frac{\mu_{i,t+1}}{\mu_t} \leq \frac{1}{F'(K_{t+1}) + 1 - \delta} < \frac{1}{F'(D_t) + 1 - \delta}.$$

Therefore, for each $t \geq 0$,

$$Q_t < \frac{1}{\prod_{s=0}^{t-1}(F'(D_s) + 1 - \delta)}.$$

Consequently, $q_0 < q(F, \delta, K_0, (\xi_t)_{t\geq 0})$.
We now have for each $i \in \{1, \dots, m\}$,

$$(r_0 + 1 - \delta)k_{i,0} + \theta^i \pi(K_0) + \xi_0 a_{i,-1} \leq c_{i,0} + k_{i,1} + (1 - a_{i,-1})q_0.$$

Case 1: if $K_1 = 0$. Then $\sum_{i=1}^{m} c_{i,1} + K_2 = \xi_1$, so $u_i'(c_{i,1}) > u_i'(\xi_1)$. FOCs of $k_{i,1}$ gives that $\frac{1}{F'(0) + 1 - \delta} \geq \frac{\beta_i u'(c_{i,1})}{u_i'(c_{i,0})}$. Hence,

$$u_i'(c_{i,0}) \geq \beta_i(F'(0) + 1 - \delta)u_i'(c_{i,1}) > \beta_i(F'(0) + 1 - \delta)u_i'(\xi_1).$$

Therefore, we have

$$c_{i,0} + k_{i,1} = c_{i,0} < (u_i')^{-1}\Big(\beta_i(F'(0) + 1 - \delta)u_i'(\xi_1)\Big).$$

Case 2: if $K_1 > 0$ then $\frac{1}{F'(K_1)+1-\delta} = \frac{q_0}{q_1+\xi_1} < \frac{q(F,\delta,K_0,(\xi_t)_{t\geq 0})}{\xi_1}$. Hence,

$$K_1 < (F')^{-1}\Big(\delta - 1 + \frac{\xi_1}{q(F,\delta,K_0,(\xi_t)_{t\geq 0})}\Big).$$

Since markets clear, we have $K_1 < D_0 - c_{i,0}$ and then

$$c_{i,1} < F(K_1) + (1-\delta)K_1 + \xi_1 < F(D_0 - c_{i,0}) + (1-\delta)(D_0 - c_{i,0}) + \xi_1.$$

Again, FOCs of $k_{i,1}$ gives that $u_i'(c_{i,0}) \geq \beta_i(F'(K_1)+1-\delta)u_i'(c_{i,1})$, therefore,

$$u_i'(c_{i,0}) > \beta_i(F'(D_0 - c_{i,0}) + 1 - \delta)u_i'\big(F(D_0 - c_{i,0}) + (1-\delta)(D_0 - c_{i,0}) + \xi_1\big).$$

We now prove that one among two Conditions (i) and (ii) in Theorem 4.2 is not true. If (i) is true then by using Lemma 4.2, we get that $c_{i,0} < x_i < D_0$, so we have

$$c_{i,0} + k_{i,1} < x_i + (F')^{-1}\Big(\delta - 1 + \frac{\xi_1}{q(F,\delta,K_0,(\xi_t)_{t\geq 0})}\Big). \tag{30}$$

Combining two cases, we have

$$(r_0 + 1 - \delta)k_{i,0} + \theta^i\pi(K_0) + \xi_0 a_{i,-1} < \max(B_1, B_2) + (1 - a_{i,-1})q(F,\delta,K_0,(\xi_t)_{t\geq 0})$$

It means that Condition (ii) in Theorem 4.2 is not true. □

4.2 Bubble on capital asset

Definition 4. *The rate of return* ρ_t *is defined by*

$$\max_i \frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})} = \frac{1}{1-\delta+\rho_t} \tag{31}$$

So, we have $1 = (1-\delta+\rho_t)\gamma_t$ and so $Q_t = (1-\delta+\rho_t)Q_{t+1}$ for each $t \geq 0$. By interating, we get

$$\begin{aligned}
1 &= (1-\delta+\rho_0)Q_1 = (1-\delta)Q_1 + \rho_0 Q_1 \\
&= (1-\delta)(1-\delta+\rho_1)Q_2 + \rho_0 Q_1 = (1-\delta)^2 Q_2 + (1-\delta)\rho_1 Q_2 + \rho_0 Q_1 \\
&= \cdots \\
&= (1-\delta)^t Q_t + \sum_{s=0}^{t-1}(1-\delta)^t \rho_t Q_{t+1}.
\end{aligned}$$

Definition 5. *We say that there is a capital asset bubble if* $1 > \sum_{s=0}^{\infty}(1-\delta)^t\rho_t Q_{t+1}$.

We can see that there is a bubble on capital asset if and only if $\lim_{t\to\infty}(1-\delta)^t Q_t > 0$. Before state the results in this section, we need the following lemma.

Lemma 4.3. *Assume that* $F'(\infty, m) < \delta$ *and* $F(k,0) = F(0,l) = 0$ *for all* $k,l \geq 0$. *At equilibrium, if* $\lim_{t\to\infty} Q_t(1-\delta)^t > 0$ *then there exists a time* t_0 *such that* $K_t = L_t = 0$ *for all* $t \geq t_0$.

Proof. See Appendix C. □

This result suggests that physical asset bubble is very dangerous because it will create a great economic recession.

Corollary 2. *Under conditions in Lemma 4.3, if there is a bubble on capital asset then there is no bubble in financial asset.*

Proof. By using Lemma 4.3, then there exists t_0 such that $K_t = L_t = 0$ for every $t \geq t_0$. We have $k_{i,t} = 1 - \lambda_{i,t} = 0$ for every $t \geq t_0$, hence $c_{i,t} + q_t a_{i,t} = (q_t + \xi_t) a_{i,t-1}$. Since $c_{i,t} > 0$ for every $t \geq 0$, we get that $a_{i,t} > 0$ for every $t > t_0$. Therefore for each $t \geq 0$, we have $\mu_{i,t} q_t = \mu_{i,t+1}(q_{t+1} + \xi_{t+1})$, hence $\gamma_{t+1} = \frac{\mu_{i,t+1}}{\mu_{i,t}}$, and consequently for each $t \geq t_0 + 1$, we have

$$Q_t = \prod_{s=1}^{t} \gamma_s = \prod_{s=1}^{t_0} \gamma_s \prod_{s=t_0+1}^{t} \gamma_s = Q_{t_0} \frac{\mu_{i,t}}{\mu_{i,t_0+1}}.$$

So, for each $t \geq t_0 + 1$, we have

$$Q_t q_t a_{i,t} = \frac{Q_{t_0}}{\mu_{i,t_0+1}} \mu_{i,t} q_t a_{i,t}.$$

Transversality condition implies that $\lim_{t\to\infty} Q_t q_t a_{i,t} = 0$. Hence

$$\lim_{t\to\infty} Q_t q_t = \lim_{t\to\infty} \sum_{i=1}^{m} Q_t q_t a_{i,t} = 0.$$

□

Proposition 4.2. *Under Conditions in Lemma 4.3 there is no bubble on capital asset in any equilibrium which $K_t > 0$ for every $t \geq 1$.*

Proof. Assume that $F'(\infty, m) < \delta$. Consider an equilibrium in which $K_t > 0$ for every $t \geq 1$. Denote $\rho_t := \frac{r_t}{p_t}$. Since $K_t > 0$, here exists i such that $k_{i,t} > 0$. FOC of K_t and $k_{i,t}$ gives that

$$\frac{1}{F'(K_t) + 1 - \delta} = \frac{1}{\rho_t + 1 - \delta} = \max_i \frac{\mu_t}{\mu_{i,t-1}} = \gamma_{t-1} = \frac{Q_t}{Q_{t-1}}.$$

Therefore $Q_{t-1} = (\rho_t + 1 - \delta) Q_t$ for every $t \geq 1$, then by iterating, we obtain

$$1 = \lim_{t\to\infty} Q_t (1-\delta)^t + \sum_{s=1}^{\infty} Q_t \rho_t (1-\delta)^{t-1}.$$

Lemma 4.3 proves that $\lim_{t\to\infty} Q_t (1-\delta)^t = 0$, and consequently

$$1 = \sum_{s=1}^{\infty} Q_t \rho_t (1-\delta)^{t-1}.$$

□

Proposition 4.3. *Consider model with exogenous labor supply. Assume that $F'(\infty) < \delta$. Then there is no bubble in capital asset at any equilibrium.*

Proof. We will prove that $\lim_{t\to\infty} Q_t (1-\delta)^t = 0$. Indeed, assume that $\lim_{t\to\infty} Q_t (1-\delta)^t > 0$ then $\lim_{t\to\infty} r_t = 0$. FOC of K_t: $r_t \geq F'(K_t) \geq F'(A) > 0$ for all t, so $\lim_{t\to\infty} r_t \geq F'(A)$, contradiction! □

5 Capital market versus financial market

In this section, we assume that labor is exogenous. We will study the role of productivity in capital sector and dividend of asset in financial sector in the productive sector.

Is it true that: if productiviy is low and dividend are high, then consumers don't invest in physical capital?

There is no simple answer for this question.

Let start by comparing the return in term of consumtion good of physical asset and financial asset. The return of physical asset is $r_{t+1}+1-\delta$, and return of financial asset is $\frac{q_{t+1}+\xi_{t+1}}{q_t}$. In a model with complete information, if real retun of physical asset is greater than real return of physical asset then no one invests in physical asset, i.e.,

Proposition 5.1. *If* $\frac{q_{t+1}+\xi_{t+1}}{q_t} \geq (F'(0)+1-\delta)$ *then consumers do not invest in physical capital, i.e.,* $K_{t+1}=0$.

Proof. Suppose that $\frac{q_{t+1}+\xi_{t+1}}{q_t} \geq (F'(0)+1-\delta)$. If $K_{t+1}>0$, then there exists $i \in \{1,\cdots,m\}$ such that $k_{i,t+1}>0$. FOC of $k_{i,t+1}$ gives us: $\max_i\{\frac{\mu_{i,t+1}}{\mu_{i,t}}\} = \frac{1}{r_{t+1}+1-\delta}$. FOC of K_{t+1} implies that $r_{t+1}=F'(K_{t+1})<F'(0)$, hence $\max_i\{\frac{\mu_{i,t+1}}{\mu_{i,t}}\} < \frac{1}{F'(0)+1-\delta}$. On the other hand, we have $\max\{\frac{\mu_{i,t+1}}{\mu_{i,t}}\} = \frac{q_t}{q_{t+1}+\xi_{t+1}}$. Therefore $\frac{q_{t+1}+\xi_{t+1}}{q_t} > (F'(0)+1-\delta)$, contradiction. □

Corollary 3. *Assume that there exists* $\xi>0$ *such that* $\xi_t \geq \xi$ *for every* $t \geq 0$ *and* $F'(0)\leq\delta$. *Then*[4] *there is an infinite sequence* $(t_n)_{n=0}^{\infty}$ *such that* $K_{t_n}=0$ *for every* $n\geq 0$.

Proof. We claim that there exists an infinite increasing sequence $(t_n)_{n=0}^{\infty}$ such that $q_{t_n}+\xi_{t_n} > q_{t_n-1}$ for every $n \geq 0$.
Indeed, if not there exists t_0 such that $q_{t+1}+\xi_{t+1} \leq q_t$ for every $t \geq t_0$. Combining with $\xi_t \geq \xi$ for every $t\geq 0$, we can easily pr that $q_{t+t_0}+t\xi \leq q_{t_0}$ for every $t \geq 0$. Let $t \to \infty$, we have $q_{t_0}=\infty$, contradiction!
Therefore, for every $n \geq 0$, $\frac{q_{t_n}+\xi_{t_n}}{q_{t_n-1}} > 1 \geq F'(0)+1-\delta$. Proposition 5.1 implies that $K_{t_n}=0$ for every $n\geq 0$. □

Economic recession: by economic recession we mean is a situation in which the aggregate capital is less than some level K.
We now state an other version[5] of Corollary 3: assume that there exists $\xi>0$ such that $\xi_t \geq \xi$ for every $t\geq 0$ and $F'(K)\leq\delta$. Then there is an infinite sequence $(t_n)_{n=0}^{\infty}$ such that $K_{t_n}\leq K$ for every $n \geq 0$.
This result says that if productivity is small, even if dividend of financial market is not high then there will be an economic recession in the future. In this case, economic recession is not from the financial market, but from the fact that the productive sector is not competitive. This result suggests that we should invest more in technology to improve the

[4]By using the same argument, we can prove this result if Assumption "$\xi_t \geq \xi > 0$ for every $t \geq 0$" is replaced by Assumption "$\sum_{t=0}^{\infty}\xi_t=\infty$".

[5]the proof of this version is exactly the same proof of Corollary 3

competitiveness of productive sector in order to avoid economic recession.

The following result shows that even if $F'(0)$ is very low, and $(\xi_t)_{t\geq 0}$ are large, the economy can invest in physical market.

Proposition 5.2. *If* $\beta_i(F'(0)+1-\delta)u_i'(\xi_{t+1}) > u_i'(\frac{F(K_t)+(1-\delta)K_t+\xi_t}{m})$ *for every* $i=1,\dots,m$ *then* $K_{t+1}>0$ *at equilibrium.*

Proof. See Appendix C. □

Proposition 5.2 shows that financial market plays an important role on capital market. Indeed, consider an equilibrium that $K_t = 0$, assume also that ξ_t is enough large then Proposition 5.2) implies that $K_{t+1} > 0$ at equilibrium. This is due to the fact that part of the financial dividend is used to buy physical capital. Brief, financial market can create a financial bubble but it can also provide capital to the capital market.

Proposition 5.2 also prove that if the productivity $F'(0)$ is high enough then $K_{t+1} > 0$ at equilibrium. But, in some cases we don't need a very high level of $F'(0)$ to ensure $K_{t+1} > 0$.

Proposition 5.3. *Assume that there exists* $t \geq 0, T \geq 1$ *such that* $\xi_t \geq \xi_{t+T}$*. If* $(F'(0)+1-\delta)\min_i \beta_i > 1$ *then at any equilibrium, there exists* $1 \leq s \leq T$ *such that* $K_{t+s} > 0$.

Proof. See Appendix C. □

Corollary 4. *Assume that there exists an infinite decreasing sequence* $(\xi_{t_n})_{n=0}^{\infty}$*, i.e.,* $\xi_{t_n} \geq \xi_{t_{n+1}}$ *for every* $n \geq 0$*. If* $(F'(0)+1-\delta)\min_i \beta_i > 1$ *then at any equilibrium there exists an infinite sequence* $(\tau_n)_{n\geq 0}$ *such that* $K_{\tau_n} > 0$ *for every* $n \geq 0$*.*

Corollary 5. *Assume that* $\xi_t = \xi > 0$ *for every* $t \geq 0$*. If* $(F'(0)+1-\delta)\min_i \beta_i > 1$ *then at any equilibrium* $K_t > 0$ *for every* $t \geq 1$*.*

This result shows that even if productivity is not so high, consumer still invests to capital asset. Note that $(F'(0)+1-\delta)\min_i \beta_i > 1$ is equivalent to $F'(0) > \frac{1}{\min_i \beta_i} - 1 + \delta$, where $\frac{1}{\min_i \beta_i} - 1 + \delta$ is the highest real return. Our result says that in the case financial dividend don't vary, if the productivity of productive sector at the origin is higher than real return then the economy always invests to the productive sector.

5.1 Some examples

Example 1. (Financial dividends are very high, $F'(0) \in (1,\infty)$, and $K_t > 0$ for every $t \geq 0$.)
Let $\xi_t = \xi > 0$ *arbitrary. There are 2 agents:* i *and* j *with* $\beta_i = \beta_j = \beta \in (0,1)$, $u_i(c) = \frac{c^{1-\sigma_1}}{1-\sigma_i}$, $u_j(c) = \frac{c^{1-\sigma_j}}{1-\sigma_j}$.
Let F *such that* $\beta(1-\delta+F'(1)) = 1$ *(this condition includes* $F'(1) > \delta$*).*

Denote $\alpha := 1-\delta+F'(1)$. *An equilibrium is given by the following: for every* $t \geq 0$,

$$\begin{aligned}
r_t &= F'(1), \quad q_t = \frac{\xi}{1-\alpha}, && (32)\\
a_{i,t} &= a_i > 0, \quad a_{j,t} = a_j > 0, && (33)\\
k_{i,t} &= k_i > 0, \quad k_{j,t} = k_j > 0, && (34)\\
c_{i,t} &= c_i = k_i(-\delta + F'(1)) + a_i\xi + \theta^i\pi && (35)\\
c_{j,t} &= c_j = k_j(-\delta + F'(1)) + a_j\xi + \theta^j\pi, && (36)
\end{aligned}$$

where $a_i + a_j = 1, k_i + k_j = 1$, $\pi = F(1) - F'(1)$.
Multipliers are given by

$$\begin{aligned}
\mu_{i,0} &:= \frac{1}{c_i^{\sigma_i}}, \quad \mu_{i,t} = \mu_{i,0}\beta^t, \quad \forall t \geq 1 && (37)\\
\mu_{j,0} &:= \frac{1}{c_j^{\sigma_i}}, \quad \mu_{j,t} = \mu_{j,0}\beta^t, \quad \forall t \geq 1. && (38)
\end{aligned}$$

The following example suggests that $(F'(0)+1-\delta)\beta_i > 1$ is necessary to ensure that $K_t > 0$.

Example 2. ($\beta(F'(0)+1-\delta) \leq 1$**, financial dividends are large, and** $K_t = 0$**, for every** $t \geq 0$**.)**
Consider an economy: there are two agents i *and* j.

$$\begin{aligned}
&\beta_i = \beta_i = \beta \in (0,1), \quad u_i(x) = u_j(x) = \frac{x^{1-\sigma}}{1-\sigma},\\
&K_0 > 0 \textit{ will be chosen }, \quad \beta(F'(0)+1-\delta) \leq 1,\\
&a_{i,-1} = \theta^i = \frac{k_{i,0}}{K_0} = a \in (0,1),\\
&\xi_0; \quad \xi_t = \xi \quad \forall t \geq 1 \textit{ will be chosen.}
\end{aligned}$$

where q_0, ξ_0, ξ, K_0 *such that*

$$\begin{aligned}
&1 \geq \beta(F'(0)+1-\delta)\Big(\frac{F(K_0)+(1-\delta)K_0+\xi_0}{\xi}\Big)^\sigma,\\
&\Big(\frac{F(K_0)+(1-\delta)K_0+\xi_0}{\xi}\Big)^\sigma = \frac{q_0}{\xi}\frac{1-\beta}{\beta}.
\end{aligned}$$

An quilibrium is given as the following

$$\begin{aligned}
\textit{Allocations:}\quad & a_{i,t} = a, \quad a_{j,t} = 1-a \quad \forall t \geq 1,\\
& k_{i,t} = k_{j,t} = 0 \quad \forall t \geq 1,\\
& c_{i,0} = a(F(K_0)+(1-\delta)K_0+\xi_0), \quad c_{i,t} = a\xi, \quad \forall t \geq 1,\\
& c_{j,0} = (1-a)(F(K_0)+(1-\delta)K_0+\xi_0), \quad c_{j,t} = (1-a)\xi, \quad \forall t \geq 1,\\
\textit{Prices:}\quad & r_0 = F'(K_0), \quad r_t = F'(0) \quad \forall t \geq 1,\\
& q_0, \quad q_t = \xi\frac{\beta}{1-\beta} \quad \forall t \geq 1.\\
\textit{Multipliers:}\quad & \mu_{i,0} = u_i'(c_{i,0}), \quad \mu_{i,0} = \beta^t u_i'(c_i)\forall \quad t \geq 1,\\
& \mu_{j,0} = u_i'(c_{j,0}), \quad \mu_{j,t} = \beta^t u_j'(c_j)\forall \quad t \geq 1.
\end{aligned}$$

Proof. See Appendix D. □

The following example suggests that even if $F'(0)$ is very small, then there is an equilibrium in which $K_1 > 0$. That is because aggregate endowment at initial date $F(K_0) + (1-\delta)K_0 + \xi_0$ are very big.

Example 3. *Consider an economy: there are two agents i and j.*

$$\begin{aligned}
&\beta_i = \beta_i = \beta \in (0,1), \quad u_i(x) = u_j(x) = \frac{x^{1-\sigma}}{1-\sigma}, \\
&K_0 > 0, K_1 \text{ will be chosen }, \quad \beta(F'(0) + 1 - \delta) \leq 1, \\
&a_{i,-1} = \theta^i = \frac{k_{i,0}}{K_0} = a \in (0,1), \\
&\xi_0, \xi_1; \quad \xi_t = \xi \quad \forall t \geq 1 \text{ will be chosen.}
\end{aligned}$$

Allocations at initial date and date 1

$$\begin{aligned}
&c_{i,0} = a(F(K_0) + (1-\delta)K_0 + \xi_0 - K_1), c_{j,0} = (1-a)(F(K_0) + (1-\delta)K_0 + \xi_0 - K_1) \\
&k_{i,1} = aK_1, \quad k_{j,1} = (1-a)K_1, \\
&a_{i,0} = a, \quad a_{j,0} = 1 - a \\
&c_{i,1} = a(F(K_1) + (1-\delta)K_1 + \xi_1), c_{j,1} = (1-a)(F(K_1) + (1-\delta)K_1 + \xi_1), \\
&K_2 = 0, \\
&a_{i,1} = a, \quad a_{j,1} = 1 - a.
\end{aligned}$$

Allocations at date t, $t \geq 2$

$$\begin{aligned}
&c_{i,t} = a\xi, \quad a_{j,t} = (1-a)\xi \\
&K_t = 0, \quad \forall t \geq 0, \\
&a_{i,t} = a \in (0,1), \quad a_{j,t} = 1 - a.
\end{aligned}$$

Prices

$$\begin{aligned}
&p_t = 1 \quad \forall t \geq 0, \\
&r_0 = F'(K_0), \quad r_1 = F'(K_1), \quad r_t = F'(0) \quad \forall t \geq 0, \\
&q_0 = (q_1 + \xi_1)\beta\Big(\frac{F(K_0) + (1-\delta)K_0 + \xi_0 - K_1}{F(K_1) + (1-\delta)K_1 + \xi_1}\Big)^{\sigma}, \\
&q_1 = \frac{\beta}{1-\beta}\xi\Big(\frac{F(K_1) + (1-\delta)K_1 + \xi_1}{\xi}\Big)^{\sigma}, \\
&q_t = \frac{\beta}{1-\beta}\xi, \forall t \geq 2.
\end{aligned}$$

We require the following conditions

$$\begin{aligned}
&\frac{1}{F'(K_0) + 1 - \delta} = \beta\Big(\frac{F(K_0) + (1-\delta)K_0 + \xi_0 - K_1}{F(K_1) + (1-\delta)K_1 + \xi_1}\Big)^{\sigma}, \text{ to choose } K_0, \\
&1 \geq \beta(F'(0) + 1 - \delta)\Big(\frac{F(K_1) + (1-\delta)K_1 + \xi_1}{\xi}\Big)^{\sigma}, \text{ to choose } K_1, \\
&\Big(\frac{F(K_1) + (1-\delta)K_1 + \xi_1}{\xi}\Big)^{\sigma} = \frac{q_1}{\xi}\frac{1-\beta}{\beta}, \text{ to choose } q_1.
\end{aligned}$$

5.2 Steady state

Definition 6. *A steady state is a sequence* $\big((c_{i,t},k_{i,t},a_{i,t})_{i=1}^m, K_t\big)_{t=0}^\infty$ *such that:*

(i) $c_{i,t}=c_i>0$, $k_{i,t}=k_i$ *for every* $i=1,\dots,m$ *and* $t\geq 0$ *and* $a_{i,t}=a_i$ *for every* $i=1,\dots,m$ *and* $t\geq -1$.

(ii) there exists a sequence of prices $(p_t,r_t,q_t)_{t=0}^\infty$ *such that* $p_t=1$ *for every* $t\geq 0$ *and* $\big((c_{i,t},k_{i,t},a_{i,t})_{i=1}^m,p_t,r_t,q_t,K_t\big)_{t=0}^\infty$ *is an equilibrium of the economy* $\mathcal{E}=\Big((u_i,\beta_i,k_{i,0},a_{i,-1})_{i=1}^m,F\Big)$.

In Example 1, we have given a steady state in the case $\beta_i=\beta$ for all $i=1,\dots,M$. The following result prove that there is a unique steady state in the case β_i are not identical.

Proposition 5.4. *Assume that* $\xi_t=\xi>0$ *for every* t*;* $\beta_1>\beta_i$ *for every* $t=2,\dots,m$ *and* $\theta^i>0$ *for every* $i=1,\dots,m$ *. Then there exists a unique steady state defined by the following*

(i) $a_1=1$, $k_1=K$ *where* K *is determined by* $\beta_1(F'(K)+1-\delta)=1$.

(ii) $k_i=a_i=0$ *for every* $i=2\dots,m$.

(iii) $c_1=(r-\delta)K+\xi+\theta^1\pi$, *where* $\pi=F(K)-rK$ *and* $r=F'(K)$*;* $c_i=\theta^i\pi$ *for every* $i=2\dots,m$.

(iv) $r_t=r$ *for every* $t\geq 0$.

(v) $q_0=\xi\dfrac{\beta_1}{1-\beta_1}$, *and* $q_{t+1}+\xi=\dfrac{1}{\beta_1}q_t$, *i.e.,* $q_t=\xi\dfrac{\beta_1}{1-\beta_1}$ *for every* $t\geq 0$.

Proof. Suppose that $\xi_t=\xi>0$ for every t; $\beta_1>\beta_i$ for every $t=2,\dots,m$ and $\theta^i>0$ for every $i=1,\dots,m$.
Let $\big((c_{i,t},k_{i,t},a_{i,t})_{i=1}^m, K_t\big)_{t=0}^\infty$ be a steady state.
If $a_i>0$ for some $i\in\{2,\dots,m\}$. FOC of $a_{i,t}$ implies that

$$\beta_i=\frac{\mu_{i,t+1}}{\mu_{i,t}}=\frac{1}{r_{t+1}+1-\delta}=\max_j\frac{\mu_{j,t+1}}{\mu_{j,t}}\geq\frac{\mu_{1,t+1}}{\mu_{1,t}}=\beta_1,$$

contradiction $\beta_1>\beta_i$. Therefore $a_i=0$ for every $i=2\dots,m$, and so $a_1=1$.
By the same argument, we get $k_1=K$ and $k_i=0$ for every $i=2,\dots,m$. Moreover, FOC of $a_{1,t}$ implies that $\beta_1(F'(K)+1-\delta)=1$ which determines K, and FOC of K_t implies that $r_t=F'(K_t)=F'(K)$. Consequently, we obtain (iii) and (iv).
By using FOC of $a_{1,t}$, we get $q_{t+1}+\xi=\dfrac{1}{\beta_1}q_t$, hen $Q_t=\beta_1^t$. Proposition **??** implies that $q_0=\sum_{t=1}^\infty Q_t\xi_t=\xi\dfrac{\beta_1}{1-\beta_1}$. □

6 Efficiency

We follow Becker, Bosi, Le Van and Seegmuller ([BBLVS11]).

Proposition 6.1. (Efficiency without $\lim_{t\to\infty}Q_t=0$**)** *An equilibrium which has the property* $\lim_{t\to\infty}Q_t(1-\delta)^t>0$ *is efficient.*

Proof. Use the same argument in Appendix 4 in Becker, Bosi, Le Van and Seegmuller ([BBLVS11]), and note that $K_t = 0$ for every $t \geq t_0$. □

Proposition 6.2. *An equilibrium in which there exists t_0 such that $\dfrac{q_{t+1} + \xi_{t+1}}{q_t} \geq (F'(0) + 1 - \delta)$ for every $t \geq t_0$ is efficient.*

Proposition 6.3. *If $\liminf_{t\to\infty} \xi_t > 0$ then the economy is efficient.*

Proof. Assume that $\liminf_{t\to\infty} \xi_t > 0$. Combining with (22), we get $\lim_{t\to\infty} Q_t = 0$, hence the economy is efficient. □

7 Conclusion

We build an infinite-horizon dynamic deterministic general equilibrium model in which heterogenous agents invest in capital or financial asset, work and consume. We proved the existence of equilibrium in this model, even if aggregate capital is not uniformly bounded.

Bubbles are studied in this framework. Consider an asset, there is a bubble on this asset if its price is greater than its fundamental value. In our model, both bubble on capital asset and bubble on financial asset may occur, but they can not occur at the same time.

Using this framework with exogeneous labor supply, we studied the relationship between financial market and capital market. Financial asset can create a bubble but it can aslo provide capital to the productive sector. The race between physical productivity and financial dividend is described as follows

- When $F'(0) < \delta$, and dividend of financial asset is not so small then capital market will disappear at infinitly many date.
- When $F'(0) < \dfrac{1}{\beta} - 1 + \delta$ we have an example in which dividend of financial asset is **high** and no investment in capital.
- When $F'(0) = \infty$ then the economy will invest at any period whatever are the dividends.
- Given $F'(0)$ finite, if ξ_t is very large w.r.t. ξ_{t+1} then $K_{t+1} > 0$.

A The existence of equilibrium in $\mathcal{E}_b^T$

Denote $\mathcal{P} := \{z_0 = (p, r, w, q) : -1 \leq p_t, q_t \leq 1, 0 \leq r_t, w_t \leq 1 \quad \forall t = 0, \dots, T\}$,

$$B_i(p, r, w, q) := \Big\{(c_i, k_i, \lambda_i, a_i) \in \mathcal{C}_i \times \mathcal{K}_i \times \Lambda_i \times \mathcal{A}_i \text{ such that}$$
$$p_t(c_{it} + k_{i,t+1} - (1-\delta)k_{i,t}) + q_t a_{i,t} < r_t k_{i,t} + (q_t + p_t\xi_t)a_{i,t-1} + w_t(1 - \lambda_{i,t}) + \gamma(p_t, r_t, w_t, q_t)$$
$$p_T(c_{i,T} - (1-\delta)k_{i,T}) < (r_T k_{i,T} + p_T\xi_T a_{i,T-1} + w_T(1 - \lambda_{i,T}) + \gamma(p_T, r_T, w_T)\Big\},$$

and

$$C_i(p,r,w,q) := \Big\{(c_i,k_i,\lambda_i,a_i) \in \mathcal{C}_i \times \mathcal{K}_i \times \Lambda_i \times \mathcal{A}_i \text{ such that}$$
$$p_t(c_{it}+k_{i,t+1}-(1-\delta)k_{i,t}) + q_t a_{i,t} \leq r_t k_{i,t} + (q_t+p_t\xi_t)a_{i,t-1} + w_t(1-\lambda_{i,t}) + \gamma(p_t,r_t,w_t,q_t)$$
$$p_T(c_{i,T}-(1-\delta)k_{i,T}) \leq r_T k_{i,T} + p_T\xi_T a_{i,T-1} + w_T(1-\lambda_{i,T}) + \gamma(p_T,r_T,w_T)\Big\},$$

where

$$\begin{aligned}
\gamma(p_0,r_0,w_0,q_0) &:= 1 - 1_{\{p_0\neq 0\}}1_{\{r_0\neq 0\}}1_{\{q_0\neq 0\}}1_{\{w_0\neq 0\}}\min\{1,|p_0|+r_0+w_0+|q_0|\} \\
\gamma(p_t,r_t,w_t,q_t) &:= 1-\min\{1,|p_t|+r_t+w_t+|q_t|\}, \quad t=1,\dots,T-1, \\
\gamma(p_T,r_T,w_T) &:= 1-\min\{1,|p_T|+r_T+w_T\}.
\end{aligned}$$

Denote $\bar{B}_i(z_0)$ is the closure of $B_i(z_0)$. We have the following result

Lemma A.1. *Given* (p_t,r_t,w_t,q_t) *such that* $-1 \leq p_t,q_t \leq 1, 0 \leq r_t,w_t \leq 1$. *If* $B_a > a_{i,t-1} > 0, B_k > k_{i,t} > 0$, *then there exists* $c_{i,t} \in [0,B_c], \lambda_{i,t} \in [0,1]$ *and* $k_{i,t+1} \in (0,B_k), a_{i,t} \in (0,B_a)$ *such that*

$$\begin{aligned}
0 \;<\; & \gamma(p_t,r_t,w_t,q_t) + r_t k_{i,t} + w_t(1-\lambda_{i,t}) + q_t(a_{i,t-1}-a_{i,t}) && (39)\\
& + p_t(\xi_t a_{i,t-1} - c_{i,t} - k_{i,t+1} + (1-\delta)k_{i,t}). && (40)
\end{aligned}$$

Proof. Choose $0 < a_{i,t} < a_{i,t-1}$, $\lambda_{i,t} < 1$. Choose $B_c > \epsilon > 0$ such that $k_{i,t+1} = \xi_t a_{i,t-1} + (1-\delta)k_{i,t} - \epsilon > 0$, choose $c_{i,t} \in [0,B_c]$ such that $sign(p_t)(\epsilon - c_{i,t}) > 0$. □

Corollary 6. *For every* $(p,r,w,q) \in \mathcal{P}$, *we have* $B_i(p,r,w,q) \neq \emptyset$ *and* $\bar{B}_i(p,r,w,q) = C_i(p,r,w,q)$.

Proof. At date 0.

Case 1: If $1_{\{p_0\neq 0\}}1_{\{r_0\neq 0\}}1_{\{q_0\neq 0\}}1_{\{w_0\neq 0\}} = 0$ then $\gamma(p_0,r_0,q_0,w_0) = 1$. We choose $a_{i,0} = a_{i,-1}+\epsilon$, $\lambda_{i,0} = \epsilon$, $k_{i,1} = \xi_0 a_{i,-1} + (1-\delta)k_{i,0} + \epsilon/2 > 0$, and $c_{i,0} = \epsilon/2$. Then for $\epsilon > 0$ is sufficient small, we have

$$p_0(c_{i,0}+k_{i,1}-(1-\delta)k_{i,0}) + q_0 a_{i,0} < r_0 k_{i,0} + (q_0+p_0\xi_0)a_{i,-1} + w_0(1-\lambda_{i,0}) + \gamma(p_0,r_0,w_0,q_0).$$

Case 2: If $1_{\{p_0\neq 0\}}1_{\{r_0\neq 0\}}1_{\{q_0\neq 0\}}1_{\{w_0\neq 0\}} = 1$, then $w_0 > 0$. We choose $a_{i,0} = a_{i,-1}+\epsilon$, $\lambda_{i,0} = \epsilon$, $k_{i,1} = \xi_0 a_{i,-1} + (1-\delta)k_{i,0} + \epsilon/2 > 0$, and $c_{i,0} = \epsilon/2$. Then for $\epsilon > 0$ is sufficient small, we have

$$\begin{aligned}
& p_0(\xi_0 a_{i,-1} + (1-\delta)k_{i,0} - c_{i,0} - k_{i,1}) + q_0(a_{i,-1}-a_{i,0}) + w_0(1-\lambda_{i,0}) \\
= \;& -p_0\epsilon - q_0\epsilon + w_0(1-\epsilon) > 0.
\end{aligned}$$

Therefore, we can choose $c_{i,0} \in (0,B_c)$, $k_{i,1} \in (0,B_k)$, $a_{i,0} \in (0,B_a)$ such that

$$p_0(c_{i,0}+k_{i,1}-(1-\delta)k_{i,0}) + q_0 a_{i,0} < r_0 k_{i,0} + (q_0+p_0\xi_0)a_{i,-1} + w_0(1-\lambda_{i,0}) + \gamma(p_0,r_0,w_0,q_0).$$

Using Lemma A.1 and induction argument, we obtain $B_i(p,r,w,q) \neq \emptyset$. Then $\bar{B}_i(p,r,w,q) = C_i(p,r,w,q)$. □

Lemma A.2. $B_i(p,r,w,q)$ *is lower semi-continuous correspondence on* $\mathcal{P}$. *And* $C_i(p,r,w,q)$ *is upper semi-continuous on* $\mathcal{P}$ *with compact convex values.*

Proof. Clearly. $\square$

We define $\Phi_0 := \mathcal{P}$, $\Phi_i := \mathcal{C}_i \times \mathcal{K}_i \times \Lambda_i \times \mathcal{A}_i$ for each $i = 1, \ldots, m$, $\Phi_{m+1} := \mathcal{K} \times \mathcal{L}$ and $\Phi := \prod_{i=0}^{m+1} \Phi_i$. An element $z \in \Phi$ is in the form $z = (z_i)_{i=0}^{m+1}$ where $z_0 := (p, r, w, q), z_i := (c_i, k_i, \lambda_i, a_i)$ for each $i = 1, \ldots, m$, $z_{m+1} = (K, L)$. We now define correspondences:

$$\begin{aligned}
\varphi_0 : \Phi &\rightarrow 2^{\Phi_0} \\
\varphi_0(z) &:= \Big\{(p', r', w', q') \in \mathcal{P} : \\
& 0 < \sum_{t=0}^{T} (p'_t - p_t)\Big[\sum_{i=1}^{m} (c_{i,t} + k_{i,t+1} - (1-\delta)k_{i,t}) - F(K_t, L_t) - \xi_t\Big] \\
& \sum_{t=0}^{T} (r'_t - r_t)(K_t - \sum_{i=1}^{m} k_{i,t}) + \sum_{t=0}^{T} (w'_t - w_t)(L_t - m + \sum_{i=1}^{m} \lambda_{i,t}) \\
& \sum_{t=0}^{T} (q'_t - q_t)(\sum_{i=1}^{m} a_{i,t} - 1)\Big\}.
\end{aligned}$$

For each $i = 1, \ldots, m$,

$$\begin{aligned}
\varphi_i : \Phi &\rightarrow 2^{\Phi_i} \\
\varphi_i(z) &:= \begin{cases} B_i(p, r, w, q) \text{ if } (c_i, k_i, \lambda_i, a_i) \notin C_i(p, r, w, q), \\ B_i(p, r, w, q) \cap \mathcal{P}_i(c_i, \lambda_i) \times \mathcal{K}_i \times \mathcal{A}_i \text{ if } (c_i, k_i, \lambda_i, a_i) \in C_i(p, r, w, q), \end{cases}
\end{aligned}$$

where $\mathcal{P}_i(c_i, \lambda_i) := \{(c'_i, \lambda'_i) : \sum_{t=0}^{T} \beta_i^t u_i(c'_{i,t}, \lambda'_{i,t}) > \sum_{t=0}^{T} \beta_i^t u_i(c_{i,t}, \lambda_{i,t})\}$.
Finally, we define

$$\begin{aligned}
\varphi_{m+1} : \Phi &\rightarrow 2^{\Phi_{m+1}} \\
\varphi_{m+1}(z) &:= \Big\{(K', L') \in \mathcal{K} \times \mathcal{L} : \\
& \sum_{t=0}^{T} (p_t F(K'_t, L'_t) - r_t K'_t - w_t L'_t) > \sum_{t=0}^{T} (p_t F(K_t, L_t) - r_t K_t - w_t L_t)\Big\}.
\end{aligned}$$

Lemma A.3. *φ_i is lower semi-continuous convex-valued correspondence for each $i = 0, 1, \ldots, m+1$.*

Proof. Clear. $\square$

Lemma A.4. *There exists $\bar{z} \in \Phi$ such that either $\varphi_i(\bar{z}) = \emptyset$ or $\bar{z}_i \in \varphi_i(\bar{z})$ for $i = 0, 1, \ldots, m+1$.*

Proof. See Gale and Mas-Colell (1975, 19979). $\square$

Take $\bar{z}$ as in Lemma A.4. By definition of φ_i, we see that $\bar{z}_i \notin \varphi_i(\bar{z})$ for all $i = 0, \ldots, m+1$, so $\varphi_i(\bar{z}) = \emptyset$ for all $i = 0, \ldots, m+1$.
Since $\varphi_0(\bar{z}) = \emptyset$ for each $i = 1, \ldots, m$, we get that: for each $i = 1, \ldots, m$, for every $(c_i, k_i, \lambda_i, a_i) \in C_i(\bar{p}, \bar{r}, \bar{w}, \bar{q})$

$$\sum_{t=0}^{T} \beta_i^t u_i(c_{i,t}, \lambda_{i,t}) \leq \sum_{t=0}^{T} \beta_i^t u_i(\bar{c}_{i,t}, \bar{\lambda}_{i,t}). \tag{41}$$

Since $\varphi_0(\bar{z}) = \emptyset$, we get that: for every $(p, r, w, q) \in \mathcal{P}$

$$\sum_{t=0}^{T}(p_t - \bar{p}_t)\Big[\sum_{i=1}^{m}(\bar{c}_{i,t} + \bar{k}_{i,t+1} - (1-\delta)\bar{k}_{i,t}) - F(\bar{K}_t, \bar{L}_t) - \xi_t\Big]$$
$$\sum_{t=0}^{T}(r_t - \bar{r}_t)(\bar{K}_t - \sum_{i=1}^{m}\bar{k}_{i,t}) + \sum_{t=0}^{T}(w_t - \bar{w}_t)(\bar{L}_t - m + \sum_{i=1}^{m}\bar{\lambda}_{i,t})$$
$$\sum_{t=0}^{T}(q_t - \bar{q}_t)(\sum_{i=1}^{m}\bar{a}_{i,t} - 1) \quad \leq 0. \tag{42}$$

Since $\varphi_{m+1}(\bar{z}) = \emptyset$, we get that: for every $(K, L) \in \mathcal{K} \times \mathcal{L}$

$$\sum_{t=0}^{T}\big(\bar{p}_t F(K_t, L_t) - \bar{r}_t K_t - \bar{w}_t L_t\big) \leq \sum_{t=0}^{T}\big(\bar{p}_t F(\bar{K}_t, \bar{L}_t) - \bar{r}_t \bar{K}_t - \bar{w}_t \bar{L}_t\big). \tag{43}$$

Therefore, for every $(K_t, L_t) \in \mathcal{K} \times \mathcal{L}$:

$$\bar{p}_t F(K_t, L_t) - \bar{r}_t K_t - \bar{w}_t L_t \leq \bar{p}_t F(\bar{K}_t, \bar{L}_t) - \bar{r}_t \bar{K}_t - \bar{w}_t \bar{L}_t. \tag{44}$$

Particular, we have $\bar{p}_t F(\bar{K}_t, \bar{L}_t) - \bar{r}_t \bar{K}_t - \bar{w}_t \bar{L}_t \geq 0$.

Lemma A.5. *For each $t = 0, \ldots, T$, if $\bar{p}_t > 0$ then $\bar{K}_t - \sum_{i=1}^{m}\bar{k}_{i,t} \geq 0$ and $\bar{L}_t - \sum_{i=1}^{m}\bar{l}_{i,t} \geq 0$.*

Proof. Assume that $p_t > 0$. If $\bar{K}_t - \sum_{i=1}^{m}\bar{k}_{i,t} < 0$. Since (42), we get $\bar{r}_t = 0$. Combining with (44), we have $\bar{K}_t = B_K$. On the other hand, $\sum_{i=1}^{m}\bar{k}_{i,t} \leq mB_k < B_K$. Contradiction! So $\bar{K}_t - \sum_{i=1}^{m}\bar{k}_{i,t} \geq 0$.
If $\bar{L}_t - \sum_{i=1}^{m}\bar{l}_{i,t} < 0$, then (42) implies $\bar{w}_t = 0$. Combining with (44), we have $\bar{L}_t = B_L$. On the other hand, $\bar{L}_t - \sum_{i=1}^{m}\bar{l}_{i,t} < 0$ implies that $\bar{L}_t < \sum_{i=1}^{m}\bar{l}_{i,t} < m$. Contradiction to $B_L > m$. So $\bar{L}_t - \sum_{i=1}^{m}\bar{l}_{i,t} \geq 0$ □

We denote

$$\bar{Z}_t := \sum_{i=1}^{m}(\bar{c}_{i,t} + \bar{k}_{i,t+1} - (1-\delta)\bar{k}_{i,t}) - F(\bar{K}_t, \bar{L}_t) - \xi_t$$
$$\bar{Y}_t := \sum_{i=1}^{m} a_{i,t} - 1.$$

Lemma A.6. *For each $t = 0, \ldots, T$*

(i) If $\bar{Z}_t \neq 0$ and $p_t \bar{Z}_t \leq \bar{p}_t \bar{Z}_t$ for every p_t with $|p_t| \leq 1$ then $|\bar{p}_t| = 1$ and $\bar{p}_t \bar{Z}_t > 0$.

(ii) If $\bar{Z}_t \neq 0$ then $\bar{Z}_t > 0$ and $\bar{p}_t = 1$. Consequently, $\bar{Z}_t \bar{p}_t \geq 0$.

(iii) $\sum_{i=1}^{m} \bar{c}_{i,t} < B_c$.

(iv) If $\bar{Y}_t \neq 0$ *then* $|\bar{q}_t| = 1$ *and* $\bar{q}_t\bar{Y}_t > 0$. *Consequently,* $Y_t q_t \geq 0$.

Proof. (i) is clear. We will give a proof for (ii). Since (42), we have $p_t\bar{Z}_t \leq \bar{p}_t\bar{Z}_t$ for every p_t with $|p_t| \leq 1$ then by using (i) we get $|\bar{p}_t| = 1$ and $\bar{p}_t\bar{Z}_t > 0$.
Assume that $\bar{Z}_t < 0$ then $\bar{p}_t = -1$. Combining with (41), we obtain $\bar{c}_{i,t} = B_c$ for all $i = 1, \ldots, m$, hence $\sum_{i=1}^{m} \bar{c}_{i,t} > B_c$.
On the other hand, by definition of $\bar{Z}_t$ we have

$$\begin{aligned} B_c < \sum_{i=1}^{m} \bar{c}_{i,t} &\leq F(\bar{K}_t, \bar{L}_t) + \xi_t + (1-\delta)\sum_{i=1}^{m} \bar{k}_{i,t} \\ &\leq F(B_K, m) + \max_{t\leq T} \xi_t + (1-\delta)mB_k \\ &\leq F(B_K, m) + \max_{t\leq T} \xi_t + (1-\delta)B_K < B_c. \end{aligned}$$

Contradiction!

(iii): If $\bar{Z}_t = 0$ then $\sum_{i=1}^{m} \bar{c}_{i,t} \leq F(B_K, m) + \max_{t\leq T} \xi_t + (1-\delta)B_K < B_c$.

If $\bar{Z}_t \neq 0$, (ii) implies that $\bar{p}_t = 1 > 0$. Lemma A.5 implies that $\sum_{i=1}^{m} \bar{k}_{i,t} \leq \bar{K}_t$ and $\sum_{i=1}^{m} \bar{l}_{i,t} \leq \bar{L}_t$. By taking the sum of budget constraint in definition of C_i, we have

$$\begin{aligned} \bar{p}_t\bar{Z}_t + \bar{q}_t\bar{Y}_t &\leq \bar{r}_t\sum_{i=1}^{m} \bar{k}_{i,t} + \bar{w}_t\sum_{i=1}^{m} \bar{l}_{i,t} - \bar{p}_tF(\bar{K}_t, \bar{L}_t) + m \\ &\leq \bar{r}_t\bar{K}_t + \bar{w}_t\bar{L}_t - \bar{p}_tF(\bar{K}_t, \bar{L}_t) \leq m, \end{aligned}$$

i.e., $\bar{Z}_t \leq m$. Consequently, we have

$$\begin{aligned} \sum_{i=1}^{m} \bar{c}_{i,t} &\leq m + F(\bar{K}_t, \bar{L}_t) + (1-\delta)\sum_{i=1}^{m} k_{i,t} + \xi \\ &\leq m + F(B_K, m) + \max_{t\leq T} \xi_t + (1-\delta)B_K < B_c. \end{aligned}$$

(iv) is proved as (ii). □

Lemma A.7. $\bar{p}_t > 0$ *for all* $t = 0, \ldots, T$.

Proof. If $\bar{p}_t \leq 0$ then $\bar{c}_{i,t} = B_c$ for all $i = 1, \ldots, m$, hence $\sum_{i=1}^{m} \bar{c}_{i,t} > B_c$, contradiction to (iii) in Lemma A.6. □

Lemma A.8. *For each* $t = 0, \ldots, T$, $\bar{r}_t > 0$ *and* $\bar{w}_t > 0$

Proof. Let $(K_t, L_t) \in \mathbb{R}^2_+$. For $\gamma \in (0, 1)$, we define $K_t(\gamma) := \gamma K_t + (1-\gamma)\bar{K}_t$ and $L_t(\gamma) := \gamma L_t + (1-\gamma)\bar{L}_t$. Since $\bar{K}_t < B_K, \bar{L}_t < B_L$, we can choose γ sufficiently close to zero such that $(K_t(\gamma), L_t(\gamma)) \in \mathcal{K} \times \mathcal{L}$. Hence, by combining (44) and the concavity of F, we have

$$\begin{aligned} \bar{p}_t F(\bar{K}_t, \bar{L}_t) - \bar{r}_t \bar{K}_t - \bar{w}_t \bar{L}_t &\geq \bar{p}_t F(\bar{K}_t(\gamma), \bar{L}_t(\gamma)) - \bar{r}_t \bar{K}_t(\gamma) - \bar{w}_t \bar{L}_t(\gamma) \\ &\geq \gamma\big[\bar{p}_t F(K_t, L_t) - \bar{r}_t K_t - \bar{w}_t L_t\big] \\ &\quad +(1-\gamma)\big[\bar{p}_t F(\bar{K}_t, \bar{L}_t) - \bar{r}_t \bar{K}_t - \bar{w}_t \bar{L}_t\big] \\ &\geq \bar{p}_t F(K_t, L_t) - \bar{r}_t K_t - \bar{w}_t L_t. \end{aligned}$$

It means that for all $(K_t, L_t) \in \mathbb{R}^2_+$

$$\bar{p}_t F(\bar{K}_t, \bar{L}_t) - \bar{r}_t \bar{K}_t - \bar{w}_t \bar{L}_t \geq \bar{p}_t F(K_t, L_t) - \bar{r}_t K_t - \bar{w}_t L_t. \tag{45}$$

If $\bar{r}_t = 0$ (or $\bar{w}_t = 0$), then we can choose $L_t > 0, K_t \to +\infty$ (or $K_t > 0, L_t \to +\infty$) then the right hand side of (45) goes to infinity, a contradiction. Therefore, we obtain $\bar{r}_t > 0$ and $\bar{w}_t > 0$. □

Lemma A.9. *For each $t = 0, \ldots, T$, $\bar{Y}_t = 0$.*

Proof. If $\bar{Y}_0 \neq 0$ then $|\bar{q}_0| = 1$ and $\bar{q}_0 \bar{Y}_0 > 0$. Noting that we have shown $\bar{p}_t, \bar{r}_t, \bar{w}_t > 0$ for all $t \geq 0$, in particular $\bar{p}_0, \bar{r}_0, \bar{w}_0 > 0$. Hence $\gamma(\bar{p}_0, \bar{r}_0, \bar{w}_0, \bar{q}_0) = 0$. By taking the sum of budget constraint in definition of C_i, we have

$$\begin{aligned} \bar{p}_0 \bar{Z}_0 + \bar{q}_0 \bar{Y}_0 &\leq \bar{r}_0 \sum_{i=1}^m \bar{k}_{i,0} + \bar{w}_0 \sum_{i=1}^m \bar{l}_{i,0} - \bar{p}_0 F(\bar{K}_0, \bar{L}_0) \\ &\leq \bar{r}_0 \bar{K}_0 + \bar{w}_0 \bar{L}_0 - \bar{p}_0 F(\bar{K}_0, \bar{L}_0) \leq 0. \end{aligned}$$

This is a contradiction with $\bar{p}_0 \bar{Z}_0 \geq 0$, $\bar{q}_0 \bar{Y}_0 > 0$. Therefore, we get $\bar{Y}_0 = 0$.
By induction argument, we conclude that $\sum_{i=1}^m \bar{a}_{i,t} = 1$ for each $t = 0, \ldots, T$. □

Lemma A.10. *For each $t = 0, \ldots, T-1$, we have $\bar{q}_t > 0$.*

Proof. We will prove by using induction argument. First, by using (iii) in Lemma A.6 we have $\bar{c}_{i,t} \leq F(B_K, m) + \max_{t \leq T} \xi_t + (1-\delta) B_K < B_c$ for each $t = 0, \ldots, T$.
We now assume that $\bar{q}_{T-1} \leq 0$, then $\bar{a}_{i,T-1} = B_a$. Indeed, if $\bar{a}_{i,T-1} < B_a$ then we can choose $\epsilon > 0$ such that $a_{i,T-1} := \bar{a}_{i,T-1} + \epsilon \in (\bar{a}_{i,T-1}, B_a)$ and $c_{i,T} := \bar{c}_{i,T} + \xi_T \epsilon \in (\bar{c}_{i,T}, B_c)$. So, this new plan violates (41). Therefore, $\bar{a}_{i,T-1} = B_a > 1$, contradiction with Lemma A.9! Therefore, we get $\bar{q}_{T-1} > 0$. By the same argument, we obtain that for each $t = 0, \ldots, T-1$, $\bar{q}_t > 0$. □

Lemma A.11. *For each $t = 0, \ldots, T$, $\bar{Z}_t = 0$.*

Proof. We will prove this lemma by induction argument. Firstly, we consider the initial period $t = 0$.
If $\bar{Z}_0 \neq 0$ then $\bar{Z}_0 > 0$ and $\bar{p}_0 = 1$. We have proved that $\bar{r}_0, \bar{w}_0, \bar{q}_0 > 0$, so $\gamma(\bar{p}_0, \bar{r}_0, \bar{w}_0, \bar{q}_0) = 0$. By taking sum of budget constraint in definition of C_i, we get

$$\begin{aligned} \bar{p}_0 \bar{Z}_0 + \bar{q}_0 \bar{Y}_0 &\leq \bar{r}_0 \sum_{i=1}^m \bar{k}_{i,0} + \bar{w}_0 \sum_{i=1}^m \bar{l}_{i,0} - \bar{p}_0 F(\bar{K}_0, \bar{L}_0) \qquad (46) \\ &\leq \bar{r}_0 \bar{K}_0 + \bar{w}_0 \bar{L}_{i,0} - \bar{p}_0 F(\bar{K}_0, \bar{L}_0) \leq 0 \quad \text{because } \bar{p}_0 > 0. \end{aligned}$$

Contradiction! because the left hand is strictly positive ($\bar{Z}_0 > 0, \bar{p}_0 = 1$, and $\bar{q}_t\bar{Y}_t \geq 0$ for every $t = 0, \ldots, T$). Hence, we have $\bar{Z}_0 = 0$.

□

Lemma A.12. *$\bar{K}_t = \sum_{i=1}^{m} \bar{k}_{i,t}$ and $\bar{L}_t = \sum_{i=1}^{m} \bar{l}_{i,t}$ for each $t = 0, \ldots, T$.*

Proof. We have shown that $\bar{p}_t > 0$, so $\bar{K}_t \geq \sum_{i=1}^{m} \bar{k}_{i,t}$. If $\bar{K}_t > \sum_{i=1}^{m} \bar{k}_{i,t}$. From (42), we have $\bar{r}_t = 1$. Hence $\gamma(\bar{p}_t, \bar{r}_t, \bar{w}_t, \bar{q}_t) = 0$. Since $(\bar{c}_i, \bar{k}_t, \bar{\lambda}_i, \bar{a}_i) \in C_i(\bar{p}, \bar{r}, \bar{w}, \bar{q})$, we get

$$\begin{aligned} 0 = \bar{p}_t\bar{Z}_t + \bar{q}_t\bar{Y}_t &\leq \bar{r}_t \sum_{i=1}^{m} \bar{k}_{i,t} + \bar{w}_t \sum_{i=1}^{m} \bar{l}_{i,t} - \bar{p}_t F(\bar{K}_t, \bar{L}_t) \\ &< \bar{r}_t\bar{K}_t + \bar{w}_t\bar{L}_{i,t} - \bar{p}_t F(\bar{K}_t, \bar{L}_t) \leq 0 \quad \text{because } \bar{r}_t = 1 > 0. \end{aligned}$$

Contradiction! Therefore $\bar{K}_t = \sum_{i=1}^{m} \bar{k}_{i,t}$. The same argument proves that $\bar{L}_t = \sum_{i=1}^{m} \bar{l}_{i,t}$. □

Lemma A.13. *$(\bar{K}_t, \bar{L}_t)$ satisfies the zero-profit condition:*

$$\bar{p}_t F(\bar{K}_t, \bar{L}_t) - \bar{r}_t\bar{K}_t - \bar{w}_t\bar{L}_t = 0 \tag{47}$$

Proof. First, we note that $\bar{K}_t = \sum_{i=1}^{m} \bar{k}_{i,t} \leq A(\xi) < B_k$ and $\bar{L}_t = \sum_{i=1}^{m} \bar{l}_{i,t} \leq m < B_l$. We knew that $\bar{p}_t F(\bar{K}_t, \bar{L}_t) - \bar{r}_t\bar{K}_t - \bar{w}_t\bar{L}_t \geq 0$. If $\bar{r}_t\bar{K}_t + \bar{w}_t\bar{L}_{i,t} - \bar{p}_t F(\bar{K}_t, \bar{L}_t) > 0$ then we can choose $(K_t, L_t) = (\mu\bar{K}_t, \mu\bar{L}_t)$ where $\mu > 1$ such that $(K_t, L_t) \in \mathcal{K} \times \mathcal{L}$. We can see that (K_t, L_t) violates (44). Contradiction! □

Lemma A.14. *For each $t = 0, \ldots, T$, $\gamma(\bar{p}_t, \bar{r}_t, \bar{w}_t, \bar{q}_t) = 0$*

Proof. $\bar{p}_t > 0$ implies that

$$\bar{p}_t(\bar{c}_{i,t} + \bar{k}_{i,t+1} - (1-\delta)\bar{k}_{i,t}) + \bar{q}_t\bar{a}_{i,t} = \bar{r}_t\bar{k}_{i,t} + (\bar{q}_t + \bar{p}_t\xi_t)\bar{a}_{i,t-1} + \bar{w}_t(1 - \bar{\lambda}_{i,t}) + \gamma(\bar{p}_t, \bar{r}_t, \bar{w}_t, \bar{q}_t).$$

By summing, we get

$$\begin{aligned} \bar{p}_t\bar{Z}_t + \bar{q}_t\bar{Y}_t &= \bar{r}_t \sum_{i=1}^{m} \bar{k}_{i,t} + \bar{w}_t \sum_{i=1}^{m} \bar{l}_{i,t} - F(\bar{K}_t, \bar{L}_t) + m\gamma(\bar{p}_t, \bar{r}_t, \bar{w}_t, \bar{q}_t) \\ &= m\gamma(\bar{p}_t, \bar{r}_t, \bar{w}_t, \bar{q}_t). \end{aligned}$$

Since $\bar{Z}_t = \bar{Y}_t = 0$, we obtain $\gamma(\bar{p}_t, \bar{r}_t, \bar{w}_t, \bar{q}_t) = 0$. □

Corollary 7. *$(\bar{p}, \bar{r}, \bar{w}, \bar{q}, (\bar{c}_i, \bar{k}_i, \bar{\lambda}_i, \bar{a}_i)_{i=1}^m, (\bar{K}, \bar{L})$ is an equilibrium of the bounded economy $\mathcal{E}_b^T$.*

B The existence of equilibrium in $\mathcal{E}$

We give a proof of Theorem 3.2.
We denote $(\bar{p}^T, \bar{r}^T, \bar{w}^T, \bar{q}^T, (\bar{c}_i^T, \bar{k}_i^T, \bar{\lambda}_i^T, \bar{a}_i^T)_{i=1}^m, \bar{K}^T, \bar{L}^T)$ is an equilibrium of $T-$ truncated economy.
We can normalize by setting $\bar{p}_t^T + \bar{r}_t^T + \bar{w}_t^T + \bar{q}_t^T = 0$ for every $t \leq T$.
By using Lemma 2.2, we see that

$$\begin{aligned}
\bar{c}_{i,t}^T &\leq \sum_{i=1}^m \bar{c}_{i,t}^T \leq D_t(K_0, \xi_0, \ldots, \xi_t) \\
\bar{k}_{i,t+1}^T &\leq \sum_{i=1}^m \bar{k}_{i,t}^T = K_t^T \leq D_t(K_0, \xi_0, \ldots, \xi_t).
\end{aligned}$$

Therefore, the sequence of equilibria $(\bar{p}^T, \bar{r}^T, \bar{w}^T, \bar{q}^T, (\bar{c}_i^T, \bar{k}_i^T, \bar{\lambda}_i^T, \bar{a}_i^T)_{i=1}^m, t\bar{K}^T, \bar{L}^T)_{T=1}^\infty$ belong to a compact set for the product topology, so we can assume that

$$\begin{aligned}
&(\bar{p}^T, \bar{r}^T, \bar{w}^T, \bar{q}^T, (\bar{c}_i^T, \bar{k}_i^T, \bar{\lambda}_i^T, \bar{a}_i^T)_{i=1}^m, (\bar{K}^T, \bar{L}^T) \\
&\xrightarrow{T\to\infty} (\bar{p}, \bar{r}, \bar{w}, \bar{q}, (\bar{c}_i, \bar{k}_i, \bar{\lambda}_i, \bar{a}_i)_{i=1}^m, (\bar{K}, \bar{L}) \quad \text{(for the product topology)}.
\end{aligned}$$

It is clear that the property (ii) in Definition 1 is hold. We will prove (i), (iii), (iv) in Definition 1.

We now write the Kuhn-Tucker necessary conditions to all variables (we can easily check that the Slater condition is satisfied). We write $x \perp y$ instead of $xy = 0$. The muliplier of variable x is denoted by $\lambda(x)$. $\bar{\mu}_{i,t}^T$ denotes the multiplier associated with budget constraints of agent i at period t. We have

$$\begin{aligned}
\bar{\mu}_{i,t}^T &\geq 0 \\
\bar{\mu}_{i,t}^T \Big[\bar{r}_t^T \bar{k}_{i,t}^T + (\bar{q}_t^T + \bar{p}_t^T \xi_t) \bar{a}_{i,t-1}^T + \bar{w}_t^T (1 - \bar{\lambda}_{i,t}^T) & \\
- \bar{p}_t^T (\bar{c}_{i,t}^T + \bar{k}_{i,t+1}^T - (1-\delta) \bar{k}_{i,t}^T) - \bar{q}_t^T \bar{a}_{i,t}^T \Big] &= 0. \qquad (48)
\end{aligned}$$

FOCs to the optimal consumption problem:

$$\begin{aligned}
\bar{c}_{i,t}^T : \quad & \beta_i^t u_i'(\bar{c}_{i,t}^T) - \bar{\mu}_{i,t}^T \bar{p}_t^T = 0 && (49) \\
\bar{\lambda}_{i,t}^T : \quad & \beta_i^t v_i'(\bar{\lambda}_{i,t}^T) - \bar{\mu}_{i,t}^T \bar{w}_t^T - \lambda(1 - \bar{\lambda}_{i,t}^T) = 0 && (50) \\
& 0 \leq \lambda(1 - \bar{\lambda}_{i,t}^T) \perp 1 - \bar{\lambda}_{i,t}^T \\
\bar{k}_{i,t+1}^T : \quad & \lambda(\bar{k}_{i,t+1}^T) - \bar{\mu}_{i,t}^T \bar{p}_t^T + \bar{\mu}_{i,t+1}^T (\bar{r}_{t+1}^T + (1-\delta) \bar{p}_{t+1}^T) = 0, && (51) \\
& 0 \leq \lambda(\bar{k}_{i,t+1}^T) \perp \bar{k}_{i,t+1}^T, \\
\bar{a}_{i,t}^T : \quad & \lambda(\bar{a}_{i,t}^T) - \bar{\mu}_{i,t}^T \bar{q}_t^T + \bar{\mu}_{i,t+1}^T (\bar{q}_{t+1}^T + \bar{p}_{t+1}^T \xi_{t+1}) = 0 && (52) \\
& 0 \leq \lambda(\bar{a}_{i,t}^T) \perp \bar{a}_{i,t}^T
\end{aligned}$$

FOCs to the optimal production problem:

$$\begin{aligned}
\bar{K}_t^T : \quad & \lambda(\bar{K}_t^T) + \bar{p}_t^T \frac{\partial F}{\partial K}(\bar{K}_t^T, \bar{L}_t^T) - \bar{r}_t^T = 0 && (53) \\
& 0 \leq \lambda(\bar{K}_t^T) \perp \bar{K}_t^T \\
\bar{L}_t^T : \quad & \lambda(\bar{L}_t^T) + \bar{p}_t^T \frac{\partial F}{\partial L}(\bar{K}_t^T, \bar{L}_t^T) - \bar{w}_t^T = 0 && (54) \\
& 0 \leq \lambda(\bar{L}_t^T) \perp \bar{L}_t^T.
\end{aligned}$$

We introduce some new variables:

$$\begin{aligned}
\bar{\zeta}^T_{i,t} &:= \beta_i^t u_i'(\bar{c}^T_{i,t})\bar{c}^T_{i,t} 1_{t\le T} && (55)\\
\bar{\eta}^T_{i,t} &:= \beta_i^t v_i'(\bar{\lambda}^T_{i,t})\bar{\lambda}^T_{i,t} 1_{t\le T} && (56)\\
\bar{\theta}^T_{i,t} &:= \beta_i^t v_i'(\bar{\lambda}^T_{i,t}) 1_{t\le T} && (57)\\
\bar{\nu}^T_{i,t} &:= \bar{\mu}^T_{i,t}\bar{w}^T_t 1_{t\le T} && (58)\\
\bar{\epsilon}^T_{i,t} &:= (\bar{\theta}^T_{i,t} - \bar{\nu}^T_{i,t}). && (59)
\end{aligned}$$

Note that $\bar{\epsilon}^T_{i,t} = \lambda(1 - \bar{\lambda}^T_{i,t})1_{t\le T}$.

Lemma B.1. *(i) For any $\epsilon > 0$, there exists τ (independently of T) such that, for any $s > \tau$, $\sum_{t=s}^{+\infty} \bar{\zeta}^T_{i,t} < \epsilon$.*

(ii) For any $\epsilon > 0$, there exists τ (independently of T) such that, for any $s > \tau$, $\sum_{t=s}^{+\infty} \bar{\eta}^T_{i,t} < \epsilon$.

Proof. We obtain the results by using the following

$$\begin{aligned}
\sum_{t=\tau}^{\infty} \beta_i^t u_i(D_t) &\ge \sum_{t=\tau}^{+\infty} \beta_i^t u_i(\bar{c}^T_{i,t}) = \sum_{t=\tau}^{\infty} \beta_i^t \Big(u_i(\bar{c}^T_{i,t}) - u_i(0)\Big)\\
&\ge \sum_{t=\tau}^{+\infty} \beta_i^t u_i'(\bar{c}^T_{i,t})\bar{c}^T_{i,t} = \sum_{t=s}^{+\infty} \bar{\zeta}^T_{i,t}. && (60)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{t=\tau}^{+\infty} \beta_i^t v_i(1) &\ge \sum_{t=\tau}^{+\infty} \beta_i^t v_i(\bar{\lambda}^T_{i,t}) = \sum_{t=\tau}^{+\infty} \beta_i^t \Big(v_i(\bar{\lambda}^T_{i,t}) - v_i(0)\Big)\\
&\ge \sum_{t=\tau}^{+\infty} \beta_i^t v_i'(\bar{\lambda}^T_{i,t})\bar{\lambda}^T_{i,t} = \sum_{t=s}^{+\infty} \bar{\eta}^T_{i,t}. && (61)
\end{aligned}$$

□

Lemma B.2. *(i) For any $\epsilon > 0$, there exists τ (independently of T) such that, for any $s > \tau$, $\sum_{t=s}^{+\infty} \bar{\nu}^T_{i,t}\bar{\lambda}^T_{i,t} < \epsilon$, $\sum_{t=s}^{+\infty} \bar{\epsilon}^T_{i,t} < \epsilon$.*

(ii) $\sum_{t=0}^{+\infty} \bar{\nu}^T_{i,t}\bar{\lambda}^T_{i,t} + \sum_{t=0}^{+\infty} \bar{\epsilon}^T_{i,t} < \dfrac{v_i(1)}{1-\beta_i}$. So $(\bar{\nu}^T_{i,t}\bar{\lambda}^T_{i,t})_{t=0}^{+\infty}$ and $(\bar{\epsilon}^T_{i,t})_{t=0}^{+\infty}$ are belong to l^1_+.

Proof. Note that

$$\begin{aligned}
\beta_i^t v_i'(\bar{\lambda}^T_{i,t})\bar{\lambda}^T_{i,t} &= \bar{\nu}^T_{i,t}\bar{\lambda}^T_{i,t} + \bar{\epsilon}^T_{i,t}\bar{\lambda}^T_{i,t}\\
&= \bar{\nu}^T_{i,t}\bar{\lambda}^T_{i,t} + \bar{\epsilon}^T_{i,t} \quad \text{since } \bar{\epsilon}^T_{i,t} \quad \text{if } \bar{\lambda}^T_{i,t} < 1.
\end{aligned}$$

Combining with (61), we get (i). (ii) is proved by taking $\tau = 0$ in (61). □

Lemma B.3. *For any* $\epsilon > 0$*, there exists* τ *such that* $\sum_{t=s}^{T} \bar{\nu}_{i,t}^T < \epsilon$ *for all* $s > \tau$*,* $T \geq s$*. In addition, for any* T

$$\sum_{t=0}^{T} \bar{\nu}_{i,t}^T < \frac{v_i(1)}{1-\beta_i} + \sum_{t=\tau}^{\infty} \beta_i^t u_i(D_t). \quad (62)$$

Proof. Since (48), we have

$$\begin{aligned}
0 &= \sum_{t=\tau}^{T} \bar{\mu}_{i,t}^T \Big[\bar{r}_t^T \bar{k}_{i,t}^T + (\bar{q}_t^T + \bar{p}_t^T \xi_t) \bar{a}_{i,t-1}^T + \bar{w}_t^T (1 - \bar{\lambda}_{i,t}^T) \\
&\quad - \bar{p}_t^T (\bar{c}_{i,t}^T + \bar{k}_{i,t+1}^T - (1-\delta)\bar{k}_{i,t}^T) - \bar{q}_t^T \bar{a}_{i,t}^T \Big] \\
&= \sum_{t=\tau}^{T} \bar{\mu}_{i,t}^T \Big[(\bar{q}_t^T + \bar{p}_t^T \xi_t) \bar{a}_{i,t-1}^T - \bar{q}_t^T \bar{a}_{i,t}^T \Big] + \sum_{t=\tau}^{T} \bar{\mu}_{i,t}^T \bar{w}_t^T (1 - \bar{\lambda}_{i,t}^T) \\
&\quad + \sum_{t=\tau}^{T} \bar{\mu}_{i,t}^T \Big[(\bar{r}_t^T + (1-\delta)\bar{p}_t^T) \bar{k}_{i,t}^T - \bar{p}_t^T \bar{k}_{i,t+1}^T \Big] - \sum_{t=\tau}^{T} \bar{\mu}_{i,t}^T \bar{p}_t^T \bar{c}_{i,t}^T \\
&= \bar{\mu}_{i,\tau}^T (\bar{q}_\tau^T + \bar{p}_\tau^T \xi_\tau) \bar{a}_{i,\tau-1}^T - \bar{\mu}_{i,T}^T \bar{q}_T^T \bar{a}_{i,T}^T + \sum_{t=\tau}^{T} \bar{\nu}_{i,t}^T - \sum_{t=\tau}^{T} \bar{\nu}_{i,t}^T \bar{\lambda}_{i,t}^T \\
&\quad + \bar{\mu}_{i,\tau}^T (\bar{r}_\tau^T + (1-\delta)\bar{p}_\tau^T) \bar{k}_{i,\tau}^T - \bar{\mu}_{i,T}^T \bar{p}_T^T \bar{k}_{i,T+1}^T - \sum_{t=\tau}^{T} \beta_i^t u_i'(\bar{c}_{i,t}^T) \bar{c}_{i,t}^T.
\end{aligned}$$

Note that $\bar{k}_{i,t+1}^T = 0$ and $\bar{a}_{i,T}^T = 0$, we obtain

$$\sum_{t=\tau}^{T} \bar{\nu}_{i,t}^T \leq \sum_{t=\tau}^{T} \bar{\nu}_{i,t}^T \bar{\lambda}_{i,t}^T + \sum_{t=\tau}^{T} \beta_i^t u_i'(\bar{c}_{i,t}^T) \bar{c}_{i,t}^T. \quad (63)$$

By combining with Lemma B.1 and Lemma B.2, we get the results. □

Lemma B.4. *Denote* $\bar{\nu}_i^T := (\bar{\nu}_{i,t}^T)_{t=0}^{+\infty}$*. There exists a subsequence* $(\bar{\nu}_i^{T_s})_{s=0}^{+\infty}$ *which converges for the* l^1*- topology to an element* $\bar{\nu}_i \in l_+^1$*. And* $\bar{\nu}_i$ *has the same properties of* $\bar{\nu}_i^T$*, i.e.,*

(i) for any $\epsilon > 0$*, there exists* τ *such that* $\sum_{t=s}^{+\infty} \bar{\nu}_{i,t} \leq \epsilon$ *for all* $s > \tau$*.*

(ii) $\sum_{t=0}^{+\infty} \bar{\nu}_{i,t} \leq \frac{v_i(1)}{1-\beta_i} + \sum_{t=\tau}^{\infty} \beta_i^t u_i(D_t)$*.*

Proof. Using Lemma 4 in Becker, Bosi, Le Van et Seegmuller ([BBLVS11]). □

Lemma B.5. *For each* $i = 1, \ldots, m$*, and* $t \geq 0$*,* $\bar{\lambda}_{i,t} > 0$*.*

Proof. We have $\bar{\theta}_{i,t}^T = \bar{\epsilon}_{i,t}^T + \bar{\nu}_{i,t}^T$ with $\bar{\epsilon}_{i,t}^T (1 - \bar{\lambda}_{i,t}^T)$. By using Lemma B.2 and (62), we have: for any $\epsilon > 0$, there exists τ such that $\sum_{t=s}^{+\infty} \bar{\theta}_{i,t}^T < \epsilon$ for any $s > \tau$.
In addition,

$$\sum_{t=0}^{+\infty} \bar{\theta}_{i,t}^T \leq \sum_{t=s}^{+\infty} \bar{\nu}_{i,t}^T + \sum_{t=s}^{+\infty} \bar{\epsilon}_{i,t}^T \leq \frac{v_i(1)}{1-\beta_i} + \sum_{t=\tau}^{\infty} \beta_i^t u_i(D_t) + \frac{v_i(1)}{1-\beta_i}.$$

Denote $\bar{\theta}_i^T := (\bar{\theta}_{i,t}^T)_{t\geq 0}$. By using Lemma 4 in Becker, Bosi, Le Van, et Seegmuller ([BBLVS11]), we have $\lim_{T\to+\infty} \bar{\theta}_i^T = \bar{\theta}_i \in l_+^1$ for the l^1-topology. Hence $\lim_{T\to+\infty} \bar{\theta}_{i,t}^T = \bar{\theta}_{i,t} < +\infty$, i.e., $\lim_{T\to+\infty} \beta_i^t v_i'(\bar{\lambda}_{i,t}^T) < +\infty$. By Inada condition, we conclude that $\bar{\lambda}_{i,t} > 0$. □

Lemma B.6. *If $\bar{p}_t > 0$ then $\bar{r}_t > 0$, $\bar{w}_t > 0$.*

Proof. Clearly since $0 = \bar{p}_t F(\bar{K}_t, \bar{L}_t) - \bar{r}_t \bar{K}_t - \bar{w}_t \bar{L}_t \geq \bar{p}_t F(K_t, L_t) - \bar{r}_t K_t - \bar{w}_t L_t$ for all $K_t, L_t \geq 0$. □

Lemma B.7. *For each $t \geq 0$, $\bar{p}_t > 0$ and hence $\bar{r}_t > 0$, $\bar{w}_t > 0$, $\bar{q}_t > 0$.*

Proof. Assume that $\lim_{T\to+\infty} \bar{p}_t^T = 0$. Since FOC of $\bar{c}_{i,t}^T$, we have $\lim_{T\to+\infty} \bar{\mu}_{i,t}^T = +\infty$ (else $\lim_{T\to+\infty} \bar{\mu}_{i,t}^T \bar{p}_t^T = 0$, hence $\beta_i^t u_i'(\bar{c}_{i,t}) = 0$, contradiction!). From FOC of $\bar{\lambda}_{i,t}^T$, we have $\bar{w}_t^T = \dfrac{\beta_i^t v_i'(\bar{\lambda}_{i,t}^T)}{\bar{\mu}_{i,t}^T} - \dfrac{\bar{\epsilon}_{i,t}^T}{\bar{\mu}_{i,t}^T}$. So, $\lim_{T\to+\infty} \bar{w}_t^T = 0$, and then $\lim_{T\to+\infty} (\bar{r}_t^T + \bar{q}_t^T) = 1$.
By using FOCs of $\bar{a}_{i,t-1}^T$ and $\bar{k}_{i,t}^T$, we get

$$\begin{aligned} \bar{\mu}_{i,t-1}^T \bar{q}_{t-1}^T + \bar{\mu}_{i,t-1}^T \bar{p}_{t-1}^T &\geq \bar{\mu}_{i,t}^T (\bar{q}_t^T + \bar{p}_t^T \xi_t) + \bar{\mu}_{i,t}^T (\bar{r}_t^T + (1-\delta)\bar{p}_t^T) \\ &\geq \bar{\mu}_{i,t}^T (\bar{q}_t^T + \bar{r}_t^T) \longrightarrow +\infty \text{ when } T \longrightarrow +\infty. \end{aligned}$$

Again by using FOCs of $\bar{a}_{i,t-2}^T$ and $\bar{k}_{i,t-2}^T$, we also get

$$\begin{aligned} \bar{\mu}_{i,t-2}^T (\bar{q}_{t-2}^T + \bar{p}_{t-2}^T) &\geq \bar{\mu}_{i,t-1}^T (\bar{q}_{t-1}^T + \bar{p}_{t-1}^T \xi_{t-1}) + \bar{\mu}_{i,t-1}^T (\bar{r}_{t-1}^T + (1-\delta)\bar{p}_{t-1}^T) \\ &\geq (1-\delta)\bar{\mu}_{i,t-1}^T (\bar{q}_{t-1}^T + \bar{p}_{t-1}^T) \longrightarrow +\infty \text{ when } T \longrightarrow +\infty. \end{aligned}$$

By induction argument, we have $\lim_{T\to+\infty} \bar{\mu}_{i,0}^T (\bar{q}_0^T + \bar{p}_0^T) = +\infty$.
If $\lim_{T\to+\infty} \bar{p}_0^T > 0$ then $\lim_{T\to+\infty} \bar{\mu}_{i,0}^T = +\infty$. Since $\lim_{T\to+\infty} \bar{\mu}_{i,0}^T \bar{w}_0^T = \lim_{T\to+\infty} \bar{\nu}_{i,0}^T < +\infty$, we get $\lim_{T\to+\infty} \bar{w}_0^T = 0$. On the other hand, using Lemma B.6, $\lim_{T\to+\infty} \bar{p}_0^T > 0$ implies that $\lim_{T\to+\infty} \bar{w}_0^T > 0$. Contradiction!
Therefore, we have $\lim_{T\to+\infty} \bar{p}_0^T = 0$. So, FOC of $\bar{c}_{i,0}^T$ gives us $\lim_{T\to+\infty} \bar{\mu}_{i,0}^T = +\infty$ for every i.
FOC of $\bar{\lambda}_{i,0}^T$ gives $\bar{\mu}_{i,0}^T \bar{w}_0^T \leq v_i'(\bar{\lambda}_{i,0}^T)$. Recall that, Lemma B.5 implies that $\lim_{T\to+\infty} \bar{\lambda}_{i,0}^T > 0$, hence $\lim_{T\to+\infty} \bar{w}_0^T = 0$. So $\bar{r}_0 + \bar{q}_0 = 1$.
The budget constraint at period 0 gives

$$\begin{aligned} &\lim_{T\to+\infty} \sum_{i=1}^m \left[\bar{p}_0^T (\bar{c}_{i,0}^T + \bar{k}_{i,1}^T - (1-\delta)\bar{k}_{i,0}^T) + \bar{q}_0^T \bar{a}_{i,0}^T \right] \\ = &\lim_{T\to+\infty} \sum_{i=1}^m \left[\bar{r}_0^T \bar{k}_{i,0}^T + (\bar{q}_0^T + \bar{p}_0^T \xi_0) \bar{a}_{i,-1}^T + \bar{w}_0^T (1 - \bar{\lambda}_{i,0}^T) \right] \end{aligned}$$

Consequently, $\bar{r}_0 \sum_{i=1}^m \bar{k}_{i,0}^T = 0$, so $\bar{r}_0 = 0$ and hence $\bar{q}_0 = 1$.
Otherwise, using the equality in the proof of Lemma B.3, we have, for every $\tau \geq 0$

$$\begin{aligned} &\bar{\mu}_{i,\tau}^T (\bar{q}_\tau^T + \bar{p}_\tau^T \xi_\tau) \bar{a}_{i,\tau-1}^T + \bar{\mu}_{i,\tau}^T (\bar{r}_\tau^T + (1-\delta)\bar{p}_\tau^T) \bar{k}_{i,\tau}^T \\ \leq &\sum_{t=\tau}^T \bar{\nu}_{i,t}^T \bar{\lambda}_{i,t}^T + \sum_{t=\tau}^T \beta_i^t u_i'(\bar{c}_{i,t}^T) \bar{c}_{i,t}^T. \end{aligned}$$

Combining with (60) and part (ii) of Lemma B.2, we get $\lim_{T\to+\infty} \bar{\mu}_{i,0}^T(\bar{q}_0^T + \bar{p}_0^T\xi_0)a_{i,-1} < +\infty$. Since $\sum_{i=1}^{m} a_{i,-1} = 1$, there is i_0 such that $a_{i_0,-1} > 0$. $\lim_{T\to+\infty} \bar{\mu}_{i_0,0}^T < +\infty$, contradiction! Therefore, we have $\bar{p}_t > 0$ and since Lemma B.6, $\bar{r}_t > 0$, $\bar{w}_t > 0$.
We finish our proof by showing that $\bar{q}_t > 0$. Firstly, note that for each t,

$$\lim_{T\to+\infty} \bar{\mu}_{i,t}^T = \lim_{T\to+\infty} \frac{\beta_i^t u_i'(\bar{c}_{i,t}^T)}{\bar{p}_t^T} \in (0, +\infty).$$

Assume there exists t such that $\lim_{T\to+\infty} \bar{q}_t^T = 0$.
FOC of $\bar{a}_{i,t}^T$: $\bar{\mu}_{i,t}^T\bar{q}_t^T \geq \bar{\mu}_{i,t+1}^T(\bar{q}_{i,t+1}^T + \bar{p}_{i,t+1}^T\xi_{t+1})$. Let $t \to +\infty$, the right-hand side is strictly positive and the left-hand is zero. Contradiction! □

It means that Condition (i) in Definition 1 is hold

Lemma B.8. *For every $t \geq 0$, $\bar{c}_{i,t} > 0$.*

Proof. Assume that $\lim_{T\to+\infty} \bar{c}_{i,t}^T = 0$. FOCs of $\bar{\lambda}_{i,t}^T$ and $\bar{c}_{i,t}^T$ give

$$\beta_i^t v_i'(\bar{\lambda}_{i,t}^T) \geq \bar{\mu}_{i,t}^T \bar{w}_t^T = \frac{\beta_i^t u_i'(\bar{c}_{i,t}^T)}{\bar{p}_t^T}\bar{w}_{i,t}^T.$$

So $\lim_{T\to+\infty} v_i'(\bar{\lambda}_{i,t}^T) = +\infty$. Contradiction to $\bar{\lambda}_{i,t} > 0$. □

Lemma B.9. *(Transversality conditions): We have*

$$\lim_{t\to+\infty} \bar{\mu}_{i,t}\bar{p}_t\bar{c}_{i,t} = 0, \quad (64)$$

$$\lim_{t\to+\infty} \bar{\mu}_{i,t}\bar{w}_t(1-\bar{\lambda}_{i,t}) = 0, \quad (65)$$

$$\lim_{t\to+\infty} \bar{\mu}_{i,t}\bar{p}_t\bar{k}_{i,t+1} = 0, \quad (66)$$

$$\lim_{t\to+\infty} \bar{\mu}_{i,t}\bar{q}_t\bar{a}_{i,t} = 0. \quad (67)$$

Proof. Let $\epsilon > 0$, we knew that there exist τ such that: for all $s' > s > \tau$

$$\sum_{t=s}^{s'} \bar{\eta}_{i,t}^T \leq \epsilon, \quad \sum_{t=s}^{s'} \bar{\nu}_{i,t}^T(1-\bar{\lambda}_{i,t}^T) \leq \epsilon. \quad (68)$$

Let $T \to +\infty$, we have

$$\epsilon \geq \lim_{T\to+\infty} \sum_{t=s}^{s'} \bar{\eta}_{i,t}^T = \sum_{t=s}^{s'} \beta_i^t u_i'(\bar{c}_{i,t}^T)\bar{c}_{i,t}^T = \sum_{t=s}^{s'} \bar{\mu}_{i,t}\bar{p}_t\bar{c}_{i,t},$$

$$\epsilon \geq \lim_{T\to+\infty} \sum_{t=s}^{s'} \bar{\nu}_{i,t}^T(1-\bar{\lambda}_{i,t}^T) = \sum_{t=s}^{s'} \bar{\mu}_{i,t}\bar{w}_t(1-\bar{\lambda}_{i,t}).$$

Hence, we get $\lim_{t\to+\infty} \bar{\mu}_{i,t}\bar{p}_t\bar{c}_{i,t} = 0$, and $\lim_{t\to+\infty} \bar{\mu}_{i,t}\bar{w}_t(1-\bar{\lambda}_{i,t}) = 0$.
On the other hand, budget constraints give

$$\begin{aligned}\sum_{t=s}^{s'} \bar{\mu}_{i,t}^T\bar{p}_t^T\bar{c}_{i,t}^T &= \sum_{t=s}^{s'} \bar{\mu}_{i,t}^T\Big[\bar{k}_{i,t}^T(\bar{r}_t^T + (1-\delta)\bar{p}_t^T) - \bar{p}_t^T\bar{k}_{i,t+1}^T + \bar{w}_t^T(1-\bar{\lambda}_{i,t}^T) \\ &\quad +(\bar{q}_t^T + \bar{p}_t^T\xi_t)\bar{a}_{i,t-1}^T - \bar{q}_t^T\bar{a}_{i,t}^T\Big] \\ \text{(using FOCs)} \quad &\geq \bar{\mu}_{i,s}^T\bar{k}_{i,s}^T(\bar{r}_s^T + (1-\delta)\bar{p}_s^T) + \bar{\mu}_{i,s}^T(\bar{q}_s^T + \bar{p}_s^T\xi_t)\bar{a}_{i,s-1}^T.\end{aligned}$$

Consequently, we have $\lim_{s\to+\infty} \bar{\mu}_{i,s}\bar{k}_{i,s}(\bar{r}_s + (1-\delta)\bar{p}_s) + \bar{\mu}_{i,s}(\bar{q}_s + \bar{p}_s\xi_t)\bar{a}_{i,s-1} = 0$.
In addition, by taking $T \to +\infty$ in (48), we have

$$\bar{\mu}_{i,t}\Big[\bar{r}_t\bar{k}_{i,t} + (\bar{q}_t + \bar{p}_t\xi_t)\bar{a}_{i,t-1} + \bar{w}_t(1-\bar{\lambda}_{i,t}) - \bar{p}_t(\bar{c}_{i,t} + \bar{k}_{i,t+1} - (1-\delta)\bar{k}_{i,t}) - \bar{q}_t\bar{a}_{i,t}\Big] = 0.$$

Combining with the results which we have just obtained, we get

$$\lim_{t\to+\infty} \bar{\mu}_{i,t}\bar{p}_t\bar{k}_{i,t+1} + \bar{\mu}_{i,t}\bar{q}_t\bar{a}_{i,t} = 0.$$

□

We now prove that Conditions (iii) and (iv) in Definition 1 are hold

Lemma B.10. *For each i, $\Big((\bar{c}_{i,t}, \bar{k}_{i,t}, \bar{\lambda}_{i,t}, \bar{a}_{i,t})_{i=1}^m\Big)_{t=0}^{\infty}$ is a solution of the problem $(P_i(\bar{p}, \bar{r}, \bar{w}, \bar{q}))$. and $(\bar{K}_t, \bar{L}_t)$ is a solution of the problem $(P(\bar{r}_t, \bar{w}_t))$.*

Proof. Assume that $(\bar{K}_t, \bar{L}_t)$ is not a solution of the problem $(P(\bar{r}_t, \bar{w}_t))$, then there exists $(K_t, L_t) \in \mathbb{R}_+^2$ such that $\bar{p}_tF(K_t, L_t) - \bar{r}_tK_t - \bar{w}_tL_t > \bar{p}_tF(\bar{K}_t, \bar{L}_t) - \bar{r}_t\bar{K}_t - \bar{w}_t\bar{L}_t$. Hence, there exists $\tau > 0$ such that: for all $T > \tau$: $\bar{p}_t^TF(K_t^T, L_t^T) - \bar{r}_t^TK_t^T - \bar{w}_t^TL_t^T > \bar{p}_t^TF(\bar{K}_t^T, \bar{L}_t^T) - \bar{r}_t^T\bar{K}_t^T - \bar{w}_t^T\bar{L}_t^T$ against the fact that $(\bar{K}_t^T, \bar{L}_t^T)$ maximizes the profit in the $T-$ truncated economy. Therefore $(\bar{K}_t, \bar{L}_t)$ is a solution of the problem $(P(\bar{r}_t, \bar{w}_t))$.
We now focus in the optimal problem of households. For each $i = 1, \ldots,$, let $(c_i, k_i, \lambda_i, a_i)$ is an alternative sequence which satisfies the budget constraints. Firstly, from budget constraints and FOCs, we have

$$\begin{aligned}&\sum_{t=0}^{T} \bar{\mu}_{i,t}(\bar{p}_tc_{i,t} + \bar{w}_t\lambda_{i,t}) \\ =\ &\sum_{t=0}^{T} \bar{\mu}_{i,t}\Big[(\bar{r}_t + (1-\delta) + \bar{p}_t)k_{i,t} + (\bar{q}_t + \bar{p}_t\xi_t)a_{i,t-1} - \bar{q}_ta_{i,t} + \bar{w}_t\Big] \\ =\ &\bar{\mu}_0(\bar{r}_0 + (1-\delta)\bar{p}_0)k_{i,0} - \bar{\mu}_T\bar{p}_Tk_{i,T+1} - \sum_{t=1}^{T} k_{i,t}\lambda(\bar{k}_{i,t}) \\ &+\bar{\mu}_0(\bar{q}_0 + \bar{p}_0\xi_0)a_{i,-1} - \bar{\mu}_T\bar{q}_Ta_{i,T} - \sum_{t=1}^{T} a_{i,t-1}\lambda(\bar{a}_{i,t-1}) + \sum_{t=0}^{T} \bar{\mu}_{i,t}\bar{w}_t.\end{aligned}$$

By applying this result and note that $a_{i,T}=0, k_{i,T+1}=0$, we have

$$\begin{aligned}
\Delta_T &:= \sum_{t=0}^{T}\beta_i^t(u_i(\bar{c}_{i,t})+v_i(\bar{\lambda}_{i,t})) - \sum_{t=0}^{T}\beta_i^t(u_i(c_{i,t})+v_i(\lambda_{i,t}))\\
&\geq \sum_{t=0}^{T}\beta_i^t u_i'(\bar{c}_{i,t})(\bar{c}_{i,t}-c_{i,t}) + \sum_{t=0}^{T}\beta_i^t v_i'(\bar{\lambda}_{i,t})(\bar{\lambda}_{i,t}-\lambda_{i,t})\\
&\geq \sum_{t=0}^{T}\bar{\mu}_{i,t}(\bar{p}_t(\bar{c}_{i,t}-c_{i,t}) + \sum_{t=0}^{T}\bar{\mu}_{i,t}(\bar{w}_t(\bar{\lambda}_{i,t}-\lambda_{i,t})\\
&= \sum_{t=0}^{T}\bar{\mu}_{i,t}(\bar{p}_t\bar{c}_{i,t}+\bar{w}_t\bar{\lambda}_{i,t}) - \sum_{t=0}^{T}\bar{\mu}_{i,t}(\bar{p}_t c_{i,t}+\bar{w}_t\lambda_{i,t})\\
&\geq -\bar{\mu}_{i,T}\bar{p}_T\bar{k}_{i,T+1} - \bar{\mu}_{i,T}\bar{q}_T\bar{a}_{i,T}.
\end{aligned}$$

Lemma B.9 implies that $\lim_{T\to+\infty}\Delta_T \geq 0$, i.e., $\Big((\bar{c}_{i,t},\bar{k}_{i,t},\bar{\lambda}_{i,t},\bar{a}_{i,t})_{i=1}^m\Big)_{t=0}^{\infty}$ is a solution of the problem $(P_i(\bar{p},\bar{r},\bar{w},\bar{q}))$. □

C Some proofs

Proof for Lemma 4.3

Because $p_t>0$ for all $t\geq 0$, we normalize by setting $p_t=1$ for all $t\geq 0$.
Assume that $F'(\infty,m)<\delta$ and $\lim_{t\to\infty}Q_t(1-\delta)^t>0$ then consumption and capital are uniformaly bounded and $\lim_{t\to\infty}Q_t=+\infty$, hence $\sum_{t\geq 0}\xi_t<+\infty$, so $\lim_{t\to\infty}\xi_t=0$. By using the same argument as in Becker, Bosi, Le Van, and Seegmuller ([BBLVS11]), we get $\lim_{t\to\infty}r_t=0$.
We are going to prove the following claims

- $\lim_{t\to\infty}L_t=0$ and $\lim_{t\to\infty}w_tL_t=0$.
- $\lim_{t\to\infty}K_t=0$ and $\lim_{t\to\infty}\sum_{i=1}^m c_{i,t}=0$, hence $\lim_{t\to\infty}w_t=0$.
- There exists t_0 such that $K_tL_t=0$, for every $t\geq t_0$, and hence $K_t=L_t=0$, because the zero-profit condition implies that if K_t then $L_t=0$.

Denote $L:=\limsup_{t\to\infty}L_t$. If $L>0$ then there exists a subsequence $(t_n)_{n\geq 1}\subset(t)_{t\geq 0}$ such that $L_{t_n}>0$ for every $n\geq 1$. We have $r_{t_n}\geq F_K(K_{t_n},L_{t_n})=F_K(\frac{K_{t_n}}{L_{t_n}},1)$. Note that $\lim_{t\to\infty}r_t=0$, we get $\lim_{n\to\infty}\frac{K_{t_n}}{L_{t_n}}=\infty$. Recall that $K_t\leq A$ for every $t\geq 0$, therefore $\lim_{n\to\infty}L_{t_n}=0$, contradiction!
So, we obtain $\limsup_{t\to\infty}L_t=0$, and hence $\lim_{t\to\infty}L_t=0$.

We now prove that $\lim_{t\to\infty}K_t=0$. Indeed, since $\lim_{t\to\infty}L_t=0$, we have $\lim_{t\to\infty}\lambda_{i,t}=0$. On the other hand, $v_i'(\lambda_{i,t})\geq w_tu_i'(c_{i,t})\geq w_tu_i'(A)$. Let $t\to\infty$, we obtain $\limsup_{t\to\infty}w_t\leq\frac{v_i'(1)}{u_i'(A)}$.

Combining with $\lim_{t\to\infty} L_t = 0$, we have $\lim_{t\to\infty} w_t L_t = 0$.
Let $\epsilon > 0$ arbitrary. There exists t_1 and $\alpha < 1$ such that $r_t + 1 - \delta \leq \alpha$ and $w_t L_t + \xi_t \leq \frac{\epsilon(1-\alpha)}{2}$ for every $t \geq t_1$. Moreover, there also exists $s > t_1$ such that $\alpha^s \leq \frac{\epsilon}{2A}$ for every $t \geq s$. Recall that $\sum_{i=1}^{m} c_{i,t} + K_{t+1} = (r_t + 1 - \delta)K_t + w_t L_t + \xi_t$ for every $t \geq 0$. Therefore, for each $t > s$

$$\begin{aligned}
K_{t_1+t} &\leq \alpha K_{t_1+t-1} + w_{t_1+t-1} L_{t_1+t-1} + \xi_{t_1+t-1} \\
&\leq \alpha\Big(\alpha K_{t_1+t-2} + w_{t_1+t-2} L_{t_1+t-2} + \xi_{t_1+t-2}\Big) + w_{t_1+t-1} L_{t_1+t-1} + \xi_{t_1+t-1} \\
&\leq \dots \\
&\leq \alpha^t K_{t_1} + \alpha^{t-1}(w_{t_1} L_{t_1} + \xi_{t_1}) + \cdots + w_{t_1+t-1} L_{t_1+t-1} + \xi_{t_1+t-1} \\
&\leq \frac{\epsilon}{2A} K_{t_1} + \frac{\epsilon(1-\alpha)}{2}\frac{1-\alpha^t}{1-\alpha} \leq \epsilon.
\end{aligned}$$

Consequently $\lim_{t\to\infty} K_t = 0$. Again, by using the fact that consumption market is clear, we obtain

$$\lim_{t\to\infty} \sum_{i=1}^{m} c_{i,t} = 0.$$

Hence, $\lim_{t\to\infty} c_{i,t} = 0$. Since $v_i'(\lambda_{i,t}) \geq w_t u_i'(c_{i,t}) \geq w_t u_i'(c_{i,t})$ and Inada condition, we imply that $\lim_{t\to\infty} w_t = 0$.
We now assume that there is an infinite subsequence $(t_n)_n \subset (t)_{t\geq 0}$ such that $K_{t_n} L_{t_n} > 0$ for every $n \geq 0$. FOC of K_{t_n} give $r_{t_n} \geq F_K(K_{t_n}, L_{t_n}) = F_K(\frac{K_{t_n}}{L_{t_n}}, 1)$. Therefore $\lim_{n\to\infty} \frac{K_{t_n}}{L_{t_n}} = \infty$, and hence $\lim_{n\to\infty} \frac{L_{t_n}}{K_{t_n}} = 0$. Consequently, $\lim_{n\to\infty} F_L(1, \frac{L_{t_n}}{K_{t_n}}) > 0$.
On the other hand, FOC of L_{t_n} implies that $w_{t_n} \geq F_L(K_{t_n}, L_{t_n}) = F_L(1, \frac{L_{t_n}}{K_{t_n}})$. Combining with $\lim_{t\to\infty} w_t =$, we get $\lim_{n\to\infty} F_L(1, \frac{L_{t_n}}{K_{t_n}}) = 0$, contradiction.
So, there exists t_0 such that $K_t L_t = 0$ for every $t \geq t_0$. If $K_t = 0$ (resp. $L_t = 0$) then from the zero-profit condition, we get that $L_t = 0$ (resp. $K_t = 0$). It means that $K_t = L_t = 0$ for every $t \geq t_0$.

Proof for Lemma 4.2

We have

$$\begin{aligned}
f_i'(x) &= u_i''(x) + \beta_i F''(D_0 - x) u_i'\Big(F(D_0 - x) + (1-\delta)(D_0 - x) + \xi_1\Big) \\
&\quad + \beta_i\Big(F'(D_0 - x) + 1 - \delta\Big)^2 u_i''\Big(F(D_0 - x) + (1-\delta)(D_0 - x) + \xi_1\Big) < 0.
\end{aligned}$$

Note that $\lim_{x\to 0^+} f(x) > 0$ because $u_i'(0) = \infty$. $\lim_{x\to D_0} f(x) = u_i'(D_0) - \beta_i\Big(F'(0) + 1 - \delta\Big) u_i'(\xi_1) < 0$. Therefore the the equation $f(x) = 0$ has a solution in $(0, D_0)$. Since $f(\cdot)$ is decreasing, the solution is unique.

Proof for Proposition 5.2

Assume that $\beta_i(F'(0)+1-\delta)u_i'(\xi_{t+1}) > u_i'(\frac{\xi_t}{m})$ for every $i=1,\dots,m$.
If $K_{t+1}=0$ then

$$\sum_{i=1}^m c_{i,t} = F(K_t)+(1-\delta)K_t+\xi_t \tag{69}$$

$$\sum_{i=1}^m c_{i,t+1}+K_{t+2} = \xi_{t+1}. \tag{70}$$

Therefore, there exists $i\in\{1,\dots,m\}$ such that $c_{i,t}\geq \frac{F(K_t)+(1-\delta)K_t+\xi_t}{m}$, so $u_i'(c_{i,t})\leq u_i'(\frac{F(K_t)+(1-\delta)K_t+\xi_t}{m})$.
On the other hand, FOC of K_{t+1} implies that $r_{t+1}\geq F'(K_{t+1})=F'(0)$. FOC of $k_{i,t+1}$ implies that $\frac{1}{r_{t+1}+1-\delta}\geq \max_j \frac{\mu_{j,t+1}}{\mu_{j,t}}$. Hence

$$\frac{1}{F'(0)+1-\delta}\geq \max_j\frac{\mu_{j,t+1}}{\mu_{j,t}} \geq \frac{\mu_{i,t+1}}{\mu_{i,t}}=\frac{\beta_i u_i'(c_{i,t+1})}{u_i'(c_{i,t})}\geq \frac{\beta_i u_i'(\xi_{t+1})}{u_i'(\frac{F(K_t)+(1-\delta)K_t+\xi_t}{m})},$$

contradicting our assumption.

Proof for Proposition 5.3

Assume that there exists $t\geq 0, T\geq 1$ such that $\xi_t\geq\xi_{t+T}$. If $(F'(0)+1-\delta)\beta_i>1$ for every $i=1,\dots,m$.
If $K_{t+s}=0$ for every $s=1,\dots,T$ then we have

$$\sum_{i=1}^m c_{i,t} = F(K_t)+(1-\delta)K_t+\xi_t,$$

$$\sum_{i=1}^m c_{i,t+s}+K_{t+s+1} = \xi_{t+s}, \quad \forall s=1,\dots,T.$$

Therefore, we have

$$\sum_{i=1}^m c_{i,t}\geq \xi_t\geq\xi_{t+T}\geq \sum_{i=1}^m c_{i,t+T}. \tag{71}$$

Consequently, there exists $i\in\{1,\cdots,m\}$ such that $c_{i,t}\geq c_{i,t+T}$, hence $u_i'(c_{i,t+T})\geq u_i'(c_{i,t})$.
On the other hand, for each $s=1,\dots,T$, FOC of K_{t+s} implies that $r_{t+s}\geq F'(K_{t+s})=F'(0)$. FOC of $k_{i,t+s}$ implies that $\frac{1}{r_{t+s}+1-\delta}\geq\max_j\frac{\mu_{j,t+s}}{\mu_{j,t+s-1}}$. Hence

$$\Big(\frac{1}{F'(0)+1-\delta}\Big)^T\geq\prod_{s=1}^T\max_j\frac{\mu_{j,t+s}}{\mu_{j,t+s-1}} \geq \prod_{s=1}^T\frac{\mu_{i,t+s}}{\mu_{i,t+s-1}}=\frac{\beta_i^T u_i'(c_{i,t+T})}{u_i'(c_{i,t})}\geq(\beta_i)^T.$$

So $1\geq(F'(0)+1-\delta)\beta_i$, contradiction!

D Examples

Proof for Example 2

Proof. It is easy to see that all markets clear and the optimal problem of firm is solved. Lets check the optimality of household's optimization problem by verifying the FOCs.

FOCs of consumption are hold since the choises of multipliers.

FOCs of $a_{h,t}$ with $h \in \{i,j\}$. We have $\frac{\mu_{h,t+1}}{\mu_{h,t}} = \beta$ for every $t \geq 1$ Since $q_t = \xi\frac{\beta}{1-\beta}$ for every $t \geq 1$, we have $\frac{q_{t+1}+\xi}{q_t} = \frac{\mu_{h,t+1}}{\mu_{h,t}}$ for every $t \geq 1$.
At initial date, we have to prove that $\frac{q_0}{q_1+\xi} = \frac{\mu_{h,1}}{\mu_{h,0}}$, i.e.,

$$\frac{q_0}{\xi}(1-\beta) = \frac{\mu_{h,1}}{\mu_{h,0}}.$$

FOC of $k_{h,t}$ with $h \in \{i,j\}$.
For $t \geq 2$, we have to prove that $\frac{1}{F'(0)+1-\delta} \geq \max_i \frac{\mu_{i,t+1}}{\mu_{i,t}}$. This is true because $\frac{\mu_{h,t+1}}{\mu_{h,t}} = \beta$ for every $t \geq 1$ and $\beta(F'(0)+1-\delta) \leq 1$.
At date 1, we have to prove that

$$1 \geq \frac{\mu_{h,1}}{\mu_{h,0}}(F'(0)+1-\delta) \quad \forall h \in \{i,j\}.$$

Therefore, we have to only check the following system

$$\begin{aligned} 1 &\geq \frac{\mu_{h,1}}{\mu_{h,0}}(F'(0)+1-\delta) \quad \forall h \in \{i,j\}. \\ \frac{q_0}{\xi}(1-\beta) &= \frac{\mu_{h,1}}{\mu_{h,0}}, \quad \forall h \in \{i,j\}. \end{aligned}$$

We have

$$\begin{aligned} \frac{\mu_{i,1}}{\mu_{i,0}} &= \beta\frac{u_i'(a\xi)}{u_i'(a(F(K_0)+(1-\delta)K_0+\xi_0))} = \beta\Big(\frac{F(K_0)+(1-\delta)K_0+\xi_0}{\xi}\Big)^{\sigma}, \\ \frac{\mu_{j,1}}{\mu_{j,0}} &= \beta\frac{u_j'(a\xi)}{u_j'(a(F(K_0)+(1-\delta)K_0+\xi_0))} = \beta\Big(\frac{F(K_0)+(1-\delta)K_0+\xi_0}{\xi}\Big)^{\sigma}. \end{aligned}$$

Our system becomes

$$\begin{aligned} 1 \geq \beta(F'(0)+1-\delta)\Big(\frac{F(K_0)+(1-\delta)K_0+\xi_0}{\xi}\Big)^{\sigma}, \\ \Big(\frac{F(K_0)+(1-\delta)K_0+\xi_0}{\xi}\Big)^{\sigma} = \frac{q_0}{\xi}\frac{1-\beta}{\beta}. \end{aligned}$$

Choose ξ, ξ_0, k_0, q_0 such that this system. □

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